Wrapped sheaves expository

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Abstract

This is the notes for the talk given in the topology seminar in University of Southern California in Fall 2022. The goal of this talk is to give a rough picture of [3] for general audience. Thus, we will focus mostly on examples and give only definitions and facts which are indispensable.

1 Introduction

The project for this talk is a continuation of a previous research by Ganatra, Pardon, and Shende in [1]. Let M be a real analytic manifold and Λ a closed subanalytic (singular) isotropic, i.e., a union of isotropic submanifolds in S^*M. The result we care from that paper is an equivalence of categories

$$\mathcal{W}(T^*M, \Lambda) \cong \mathrm{Sh}_\Lambda(M)^c.$$

We will give more information about these categories later. For the moment, we just mention that $\mathcal{W}(T^*M, \Lambda)$ is a defined through Floer theory while $\mathrm{Sh}_\Lambda(M)^c$ can be understood combinatorially.

One fact to know is that this equivalence is not proven by defining a functor and then proving that it is indeed an equivalence. Instead, the authors of [1] find a class of generators on both sides and show that these generators match functorially an hence prove the equivalence. Now, there is an equivalence established earlier in [5, 4] between the infinitesimal Fukaya category of T^*M and the category of constructible sheaves on M. It is expected that all the categories mentioned above can be fit into the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Fuk}_\epsilon(T^*M)^{op} \supseteq & \mathrm{Fuk}_\epsilon(T^*M,\Lambda)^{op} & \overset{l_{\mathrm{Fuk}}}{\dashrightarrow} & \mathcal{W}(T^*M,\Lambda)^{op} & & \\
\Vert\ [5,\ 4] & \Vert\ [5,\ 4] & & \updownarrow & \overset{[1]}{\searrow\!\!=} & \\
\mathrm{Sh}_{\mathrm{constr}}(M) \supseteq & \mathrm{Sh}_{\mathrm{constr},S^*M\setminus\Lambda}(M) & \xrightarrow{l_{\mathrm{Sh}}} & \mathfrak{w}\mathrm{sh}_\Lambda(M) & \xrightarrow{\mathfrak{W}_\Lambda^+} & \mathrm{Sh}_\Lambda(M)^c
\end{array}$$

That is, it is expected the categories considered by Ganatra, Pardon, and Shende can be obtained by localizations from those considered by Nadler and Zaslow. What is done in [3] is to define the localized category $\mathfrak{w}\mathrm{sh}_\Lambda(M)$ and a comparison functor $\mathfrak{W}_\Lambda^+$, which exhibits that it's the same as the more classical category $\mathrm{Sh}_\Lambda(M)^c$

The plan for this talk is to provide descriptions of what those categories are, in the order of $\mathcal{W}(T^*M,\Lambda)$, $\mathrm{Sh}_\Lambda(M)^c$, and finally $\mathfrak{w}\mathrm{sh}_\Lambda(M)$, as well as the functor $\mathfrak{W}_\Lambda^+$. The last two construction depends heavily from the sheaf quantization construction by Guillermou, Kashiwara, and Schapira in [2]. To make the talk simple, we use $\mathbb{Z}$ as our coefficient.

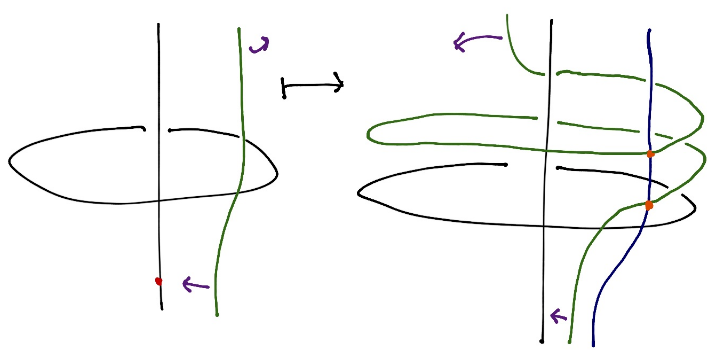
2 Wrapped Fukaya categories

Without going into too much details, we mention that the wrapped Fukaya category $\mathcal{W}(T^*M,\Lambda)$ is an A_∞-category, which means in addition to composition there are higher morphisms encoding the associativity of composition, and all such operation comes from pseudo-holomorphic disk counting.

We will only look at its homotopy category, i.e., we look at its morphisms after passing to cohomology. Now the basic objects of $\mathcal{W}(T^*M,\Lambda)$ are given Lagrangians with structures as usual. To have a well-functioning Floer theory, a Lagrangian L should be compact horizontally and conic, i.e., invariant with respect to the scaling of T^*M, at ∞. This way, we can talk about the corresponding Legendrian $\partial_\infty L$ in S^*M and we require it to be away from Λ.

For such Lagrangians L, K, we denote the morphisms between them by $\mathcal{W}(L,K)$. Then $H^*\mathcal{W}(L,K) \cong HW^*(L,K) \coloneqq \operatorname*{colim}_{K\to K^w} HF^*(L,K^w)$. Here, HF^* is the ordinary Floer cohomology define as chain complexes generated by Lagrangian intersections with differential given by disk counting. The focus today is the (positive) wrapping $K \to K^w$, which means a isotopy K_t between K and K^w such that K_t are all conic near infinity, $\partial_\infty K_t$ is an isotopy of Legendrians, and $\alpha(\partial_t\partial_\infty K_t) \geq 0$.

We illustrate it with an example: Consider $M = S^1$, $\Lambda = S^*_{0,\leq}S^1$, the negative part of the fiber at the origin, and we would like to understand the Lagrangian $L = T^*_\epsilon S^1$ for some small ϵ. We draw in the following picture how a wrapping in this situation look like:

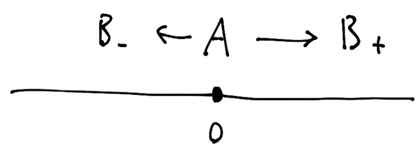


One can see that for each wrapping, the generator are given by the number of times where L goes around the circle once and hit itself. It is an exercise that no disk with correct index goes between them so $HW^*(L,L) = \operatorname*{colim}_{n\to\infty} \mathbb{Z}^{\oplus n} = \mathbb{Z}^{\mathbb{Z}_{\geq 0}}$. We mention that a better way to express it is $HW^*(L, L) = \mathbb{Z}[t]$, the polynomial rings, and if we change Λ to the whole fiber $S^*_0 S^1$ or no fiber, then the same Lagrangians gives $\mathbb{Z}$ or $\mathbb{Z}[t, t^{-1}]$.

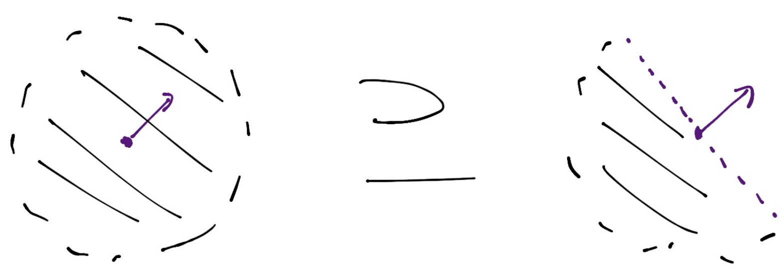
3 Sheaves with a fixed microsupport condition

We turn our attention to $\mathrm{Sh}_\Lambda(M)$. We first recall that a sheaf $F \in \mathrm{Sh}(M)$ consists of assignments $U \mapsto F(U)$, $(V \subseteq U) \mapsto F(U) \to F(V)$, and more, plus a gluing condition. For each sheaf F, there exists a conic, closed, coisotropic set $\mathrm{SS}(F)$ in T^*M, called the microsupport. On the zero section it recovers the support $\mathrm{supp}(F) = \overline{\{x | F_x \neq 0\}}$ and away from the zero section this set roughly encodes the codirections where the restriction is not an isomorphism. In general, it is hard to compute $\mathrm{SS}(F)$ but when F is constructible, meaning that there is a stratification $\mathcal{S}$ such that $F|_{X_\alpha} \in \mathrm{Loc}(X_\alpha)$ for each strata X_α in $\mathcal{S}$, $\mathrm{SS}(F)$ is easier to compute and is always a singular Lagrangian if we know that $\mathrm{SS}(F)$ is subanalytic.

Consider the one dimensional case below and call it F:



Away from 0, the sheaf is just constant so there is no microsupport. To answer whether $(0, -1)$ is in the $\mathrm{SS}(F)$, we consider the restriction $F((-1, 1)) \to F((0, 1))$, or whether the map $A \to B_+$ is an isomorphism, and similar for $(0, 1)$. In general, for a given covector, we roughly consider the following picture:



The symbol $\mathrm{Sh}_\Lambda(M)$ then stands for the subcategory of $\mathrm{Sh}(M)$ whose objects F satisfies $\mathrm{SS}^\infty(F) \subseteq \Lambda$. One property of subanalytic Legendrians Λ in S^*M is that there is always a (Whitney) triangulation $\mathcal{S}$ so that $\Lambda \subseteq S^*\mathcal{S} := \cup_{\alpha \in \mathcal{S}} N^*_\infty X_\alpha$ so $\mathrm{Sh}_\Lambda(M)$ can be seen as a subcategory of $\mathcal{S}$-Mod where some arrows are required to be isomorphisms.

Consider again the example $M = S^1$, $\Lambda = S^*_{0,\leq} S^1$. We provide directly the answer that the sheaf F corresponds to $L = T^*_\epsilon S^1$ is given by the constructible sheaves F whose stalks are given by $\mathbb{Z}^{\mathbb{Z}_{\geq 0}}$, and near 0 where the picture has the from $B_{\leftarrow} A \to B_+$, is given by

$$\mathbb{Z}^{\mathbb{Z}_{\geq 0}} = \mathbb{Z}^{\mathbb{Z}_{\geq 0}} \xrightarrow{m} \mathbb{Z}^{\mathbb{Z}_{\geq 0}}$$

where m is the shifting $m(a_0, a_1, a_2, \cdots) = (0, a_0, a_1, \cdots)$. This sheaf has an alternative expression by $F = \pi_! \mathbb{Z}_{(0,\infty)}$, where $\pi : \mathbb{R}^1 \to S^1$ is the universal cover, so a computation of the self-Hom can look like

$$\begin{aligned}
\mathrm{Hom}(\pi_!\mathbb{Z}_{(0,\infty)}, \pi_!\mathbb{Z}_{(0,\infty)}) &= \mathrm{Hom}(\mathbb{Z}_{(0,\infty)}, \pi^!\pi_!\mathbb{Z}_{(0,\infty)}) \\
&= \mathrm{Hom}(\mathbb{Z}_{(0,\infty)}, \pi^*\pi_!\mathbb{Z}_{(0,\infty)}) \\
&= \Gamma((0,\infty), \pi^*\pi_!\mathbb{Z}_{(0,\infty)}) \\
&= \left(\pi^*\pi_!\mathbb{Z}_{(0,\infty)}\right)_\epsilon \\
&= \Gamma_c(\{\epsilon + n | n \in \mathbb{Z}_{\geq 0}\}; \mathbb{Z}) = \mathbb{Z}^{\mathbb{Z}_{\geq 0}}.
\end{aligned}$$

4 Wrapped sheaves

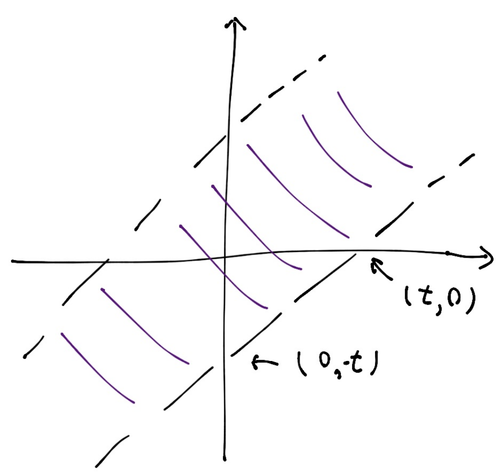
Let X, Y be topological spaces. A sheaf $K \in \mathrm{Sh}(X \times Y)$ produces a functor

$$\begin{aligned}
\mathrm{Sh}(X) &\to \mathrm{Sh}(Y) \\
F &\mapsto K \circ F := p_{2!}(K \otimes p_1^* F)
\end{aligned}$$

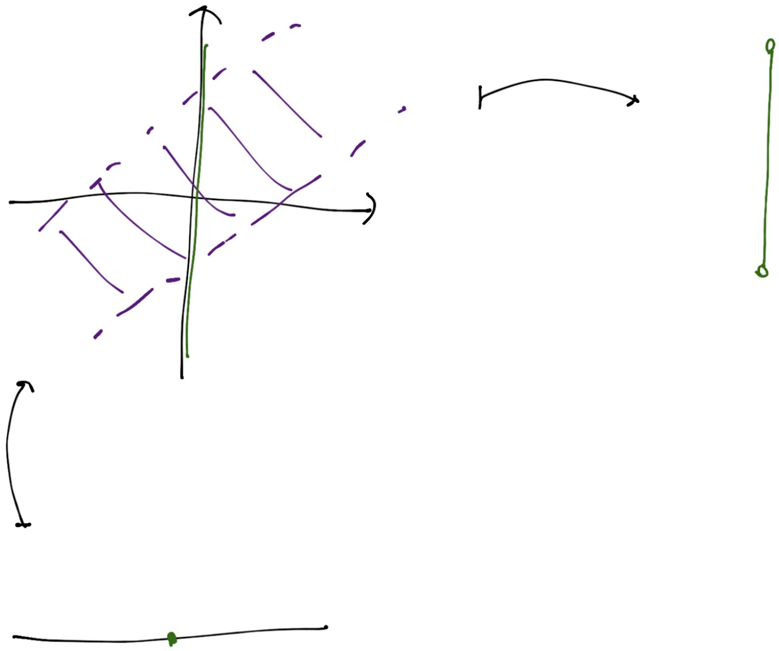
which is usually referred as convolution with the sheaf kernel K.

Now let $\varphi : S^*M \times I \to S^*M$ be a contact isotopy where I is an open interval containing 0. A theorem of Guillermou, Kashiwara, and Schapira is that there exists a unique sheaf $K = K(\varphi) \in \mathrm{Sh}(M \times M \times I)$ such that $K|_0 = \mathbb{Z}_\Delta$ and $\mathrm{SS}^\infty(K)$ is contained in the movie of φ. By convolution, we get, for a fixed sheaf $F \in \mathrm{Sh}(M)$, a family $\{F_t\}_{t \in I}$ such that $F_0 = F$ and $\mathrm{SS}^\infty(F_t) = \varphi_t \,\mathrm{SS}^\infty(F)$, and we can think it as isotope F by φ. Furthermore, when φ is positive, then there is a continuation map $K_s \to K_t$ when $t \geq s$.

As a example, consider the isotopy on $S^*\mathbb{R}^1 \cong \mathbb{R}^1 \times \pm 1$ which is given by the formula $\varphi(x, \pm 1) = (x \pm t, \pm 1)$. Its GKS sheaf quantization has $\mathbb{Z}_{\{(x,y) | |x-y| < t\}}[1]$ as the slice at $t > 0$:



It sends $F = \mathbb{Z}_{\{0\}}$, by convolution, to $\mathbb{Z}_{(-t,t)}[1]$:



Now, we define the category of wrapped sheaves. First, take $\widetilde{\mathfrak{wsh}}_\Lambda(M)$ to be the collection of sheaves F such that $\mathrm{SS}^\infty(F)$ is a subanalytic Legendrian away from Λ, $\mathrm{supp}(F)$ is compact, and F_x to be perfect for all $x \in M$.

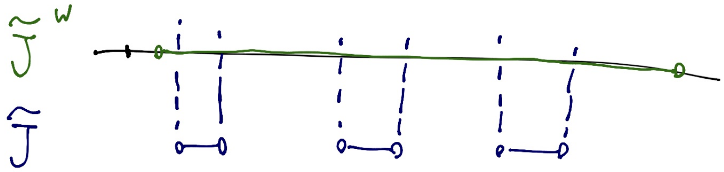
Definition 4.1. The category of wrapped sheaves $\mathfrak{wsh}_\Lambda(M)$ is defined to be

$$\mathfrak{wsh}_\Lambda(M) \coloneqq \widetilde{\mathfrak{wsh}}_\Lambda(M)/\text{isotopies}.$$

Similarly to the wrapped Fukaya category, morphisms in $\mathfrak{wsh}_\Lambda(M)$ is computed by the colimit

$$\mathrm{Hom}_w(G, F) = \operatorname*{colim}_{F \to F^w} \mathrm{Hom}(G, F^w).$$

As one can guess, we consider yet again the case $M = S^1$ and $\Lambda = S^*_{0,\leq} S^1$. It's now not hard to convince oneself that the object corresponds to $L = T^*_\epsilon S^1$ is $\mathbb{Z}_{\{0\}}$. So by the picture above, we consider $\mathrm{Hom}_w(\mathbb{Z}_J, \mathbb{Z}_J) = \operatorname*{colim}_w \mathrm{Hom}(\mathbb{Z}_J, (\mathbb{Z}_J)^w)$ where $J \hookrightarrow S^1$ is some small open interval. Depending on how large the wrapping w is, the situation in the universal cover is given by the following picture:



So $\mathrm{Hom}(\mathbb{Z}_J, (\mathbb{Z}_J)^w) = \mathbb{Z}^{\oplus n}$ where n is the number of times when the lift of J passes over itself after extended by w. Thus, we conclude again that $\mathrm{Hom}_w(\mathbb{Z}_{\{0\}}, \mathbb{Z}_{\{0\}}) = \mathbb{Z}^{\mathbb{Z}_{\geq 0}}$.

Finally, we mention that the comparison map is given by

$$\begin{aligned} \mathfrak{W}_\Lambda^+ : \mathfrak{w}\mathrm{sh}_\Lambda(M) &\to \mathrm{Sh}_\Lambda(M) \\ F &\mapsto \operatorname*{colim}_{F \to F^w} F^w, \end{aligned}$$

That is, we take the colimit directly on the objects. One can conclude, from the last picture that, in the case $M = S^1$ and $\Lambda = S^*_{0,\leq} S^1$, $\mathfrak{W}_\Lambda^+ \mathbb{Z}_{\{0\}} = \pi_! \mathbb{Z}_{(0,\infty)}$, which is the reason for our guess earlier.

References

[1] Sheel Ganatra, John Pardon, and Vivek Shende. Microlocal Morse theory of wrapped Fukaya categories. *arXiv:1809.08807v2*, 2020.

[2] Stéphane Guillermou, Masaki Kashiwara, and Pierre Schapira. Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems. *Duke Math. J.*, 161(2):201–245, 2012.

[3] Christopher Kuo. Wrapped sheaves. *arXiv:2102.06791*, 2021.

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[5] David Nadler and Eric Zaslow. Constructible sheaves and the Fukaya category. *J. Amer. Math. Soc.*, 22(1):233–286, 2009.