INTRODUCTION TO MICROLOCAL SHEAF THEORY
ORAL EXAM QUESTION LIST

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General Information

This is a draft version of the oral exam problem list; some minor modifications may be made before the end of July 12, though they will not be drastic. The oral exam will be twenty minutes long. At least three-quarters of the questions will be taken from the lists in this document. Unless specified, all topological spaces will be locally compact Hausdorff and finite dimensional. When discussing microlocal sheaf theory, we will further assume that they have the structure of a C^∞-manifold.

Topic 1: Higher category theory

(1) What is the definition of a κ-filtered category? What is the definition of a presentable category?
(2) Why do we care about presentable categories? (Recall a certain useful theorem that needs this setting.)
(3) Explain, 1-categorically, why the category of abelian groups is an accessible category.
(4) What is the definition of a stable category?
(5) How do I know that Ani is not pointed? How do I know that Ani_* is not stable?
(6) Explain why the internal Hom $\underline{\mathrm{Hom}}(x, y)$ always exists for $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{st})$.
    (a) Is it true that the canonical map $\underline{\mathrm{Hom}}(x, 1_{\mathcal{V}}) \otimes y \to \underline{\mathrm{Hom}}(x, y)$ is always an isomorphism? Hint: Consider the case $\mathcal{V} = \mathrm{Vect}_k$, k a field of characteristic 0.
    (b) What if we assume x is dualizable? (You need to recall what dualizable means.)

Topic 2: Six-functor formalism

(1) When building a six-functor formalism, we employ the notion of a "suitable decomposition". What is this decomposition and what is its significance?
(2) Recall the suitable decomposition that we choose for the category of locally compact Hausdorff spaces with continuous maps. Sketch the proof of the decomposition part showing why it is a suitable decomposition.
(3) State what the projection formula is. Give a sketch of the proof that the base change formula implies the projection formula.
(4) On the level of objects, explain what Verdier duality does. (You will need to explain what a compactly supported section is.)
(5) How is Verdier duality used to establish the six-functor formalism?

Topic 3: Microlocal sheaf theory

(1) Assume we know that anima-valued sheaves have enough points. Sketch the proof that $\mathcal{C}$-valued sheaves have enough points if $\mathcal{C}$ is ω-accessible presentable.

(2) Assume $\mathcal{C}$ is stable and ω-accessible presentable. Sketch the proof of the claim that $\mathrm{supp}(F) = \{x \mid F_x \neq 0\}$ for $F \in \mathrm{Sh}(X;\mathcal{C})$.
(3) Define the standard exact symplectic structure of a cotangent bundle T^*M.
(4) Give an example of a symplectic isotopy that is not homogeneous.
(5) Consider the homogeneous symplectic isotopy $\Phi : \dot{T}^*\mathbb{R} \times (-10, 10) \to \dot{T}^*\mathbb{R}$ that is given by

$$\Phi(x,\xi,t) = \begin{cases} (x+t,\xi), & \xi > 0, \\ (x-t,\xi), & \xi < 0. \end{cases}$$

Up to a shift, what is the time-t slice of the GKS kernel $\Phi(K)$? Verify the microsupport condition.

Topic 4: Six-functor computation

For this part of the question, you're free to assume that the coefficient $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega,st})$ is a symmetric monoidal ω-accessible presentable stable category, and you're free to use any properties of the six-functor formalism.

(1) View $\mathbb{Z}$ as a topological space, and let $a : \mathbb{Z} \to \{*\}$ be the projection.
  (a) Show that there is a canonical isomorphism $1_{\mathbb{Z}} \xrightarrow{\sim} \prod_{n\in\mathbb{Z}} 1_{\{n\}}$ and use it to compute $\Gamma(\mathbb{Z}; 1_{\mathbb{Z}})$.
  (b) What is $\Gamma_c(\mathbb{Z}; 1_{\mathbb{Z}})$?
(2) Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. Which sheaf kernel corresponds to f^*?
(3) (a) Let $i : \mathbb{R}^1 \hookrightarrow \mathbb{R}^2$ be given by $x \mapsto (x, 0)$, and show that $i^! 1_{\mathbb{R}^2} = 1_{\mathbb{R}^1}[1]$. Hint: consider the projection $p : \mathbb{R}^2 \to \mathbb{R}^1$.
  (b) Bonus: Let $\mathbb{H}$ be an open set with smooth boundary. How would you compute $\Gamma_U(1_M)$?
(4) Let $i : Z \hookrightarrow M$ be a closed subset, and recall $F_Z \coloneqq i_! i^* F$ and $\Gamma_Z(F) \coloneqq i_* i^!(F)$.
  (a) Prove that $F_Z = 1_Z \otimes F$.
  (b) Prove that if $\mathrm{supp}(F) \subseteq Z$, then $F \to F_Z$ is an isomorphism. Hint: Check at stalks.
  (c) Bonus: Is it true that $\Gamma_Z(F) = \Gamma_Z(1_M) \otimes F$? (Hint: What if $\mathrm{supp}(F) \subseteq Z$?)