

04.13.2026

- Motivation: Why sheaf

$E$  rank  $k$  holomorphic vector bundle

↓

$X$ :  $n$ -dimensional complex manifold.

Want to know  $\Gamma(X; E)$ : the vector space of holomorphic sections.

Locally on some small  $U \subseteq X$ ,  $E|_U \cong U \times \mathbb{C}^k$

so can pick a basis  $(e_1, \dots, e_n)$  s.t.

$$\Gamma_{\Delta}(U; E|_U) \cong \mathcal{O}_U^n$$

$$a_1 e_1 + \dots + a_n e_n \longmapsto (a_1, \dots, a_n).$$

To get  $\Gamma(X; E)$ , closed try to cover  $X$  by  $\{U_\alpha\}_{\alpha \in I}$

Observation: If  $S_1, S_2 \in \Gamma(X; E)$  and

$$S_1|_{U_\alpha} = S_2|_{U_\alpha} \quad \forall \alpha \in I, \text{ then } S_1 = S_2.$$

Conclusion 1:  $0 \rightarrow \Gamma(X; E) \rightarrow \prod_{\alpha \in I} \Gamma(U_\alpha; E|_{U_\alpha})$

More concretely, if  $E$  is trivial over  $U_\alpha$ ,

$$\text{then } \Gamma(X; E) \hookrightarrow \prod_{\alpha \in I} \bigoplus^k \mathbb{C}.$$

But we could say more. Set  $U_{\alpha\beta} := U_\alpha \cap U_\beta$ .

$$\begin{array}{ccccc} & & \Gamma(U_\alpha; E|_{U_\alpha}) & & \\ & \nearrow & & \searrow & \\ \Gamma(X; E) & & & & \Gamma(U_{\alpha\beta}; E|_{U_{\alpha\beta}}) \\ & \searrow & & \nearrow & \\ & & \Gamma(U_\beta; E|_{U_\beta}) & & \end{array}$$

$\curvearrowright$

So  $s \in \mathcal{P}(X; E)$  produces a family

$$(S_\alpha) \in \prod_\alpha \mathcal{P}(U_\alpha; E|_{U_\alpha}) \text{ s.t. } S_\alpha|_{U_\alpha \cap U_\beta} = S_\beta|_{U_\alpha \cap U_\beta}.$$

Observation: As sections are functions  $\begin{array}{c} E \\ \downarrow \\ X \end{array} \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} s$

any such family glues to a section  $s \in \mathcal{P}(X; E)$

$$\text{s.t. } s|_{U_\alpha} = S_\alpha.$$

Conclusion 2:  $\text{Im}(\mathcal{P}(X; E) \hookrightarrow \prod_\alpha \mathcal{P}(U_\alpha; E|_{U_\alpha}))$

$$\simeq \ker\left(\prod_\alpha \mathcal{P}(U_\alpha; E|_{U_\alpha}) \longrightarrow \prod_{\alpha, \beta} \mathcal{P}(U_{\alpha\beta}; E|_{U_{\alpha\beta}})\right)$$

$$(S_\alpha) \longmapsto (S_\alpha|_{U_{\alpha\beta}} - S_\beta|_{U_{\alpha\beta}}).$$

$\mathcal{O}_{p_x} :=$  the poset of opens in  $X$

Def<sup>n</sup>: A sheaf of abelian groups is a functor

$F: \mathcal{O}_{p_x}^{\text{op}} \rightarrow \text{Ab}$  s.t. for any

cover  $\{U_\alpha\} \downarrow U$ , we have the sequence

$$0 \rightarrow \Gamma(U; F) \rightarrow \prod_{\alpha} \Gamma(U_\alpha; F) \rightarrow \prod_{\alpha, \beta} \Gamma(U_{\alpha\beta}; F)$$

to be exact.

That is, a sheaf is an assignment that satisfies the "local-to-global" principle.

Notation:  $\text{Sh}(X) :=$  the category of sheaves on  $X$ .

$\text{Hom}(G, F) =$  natural transformation  $G \rightarrow F$ .

$$\Leftrightarrow \begin{array}{ccc} G(U) & \rightarrow & F(U) \\ \downarrow & \supseteq & \downarrow \\ G(V) & \rightarrow & F(V) \end{array} \quad \text{for } V \subseteq U.$$

It's a good tool: E.g. Cartan's proof of  
de Rham's theorem:

Thm: For a manifold  $M$ , the map

$$\int_{[\sigma]} (-) : H_{\text{dR}}^k(M) \longrightarrow H_{\text{sing}}^k(M; \mathbb{R})$$

$$[w] \mapsto \left( [\sigma] \mapsto \int_{\sigma} w \right)$$

is an isom.

"proof": Check

$$\Omega_M^i(U) := C^\infty(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \rightarrow \dots \quad \&$$

$$C_{0,1,2}^i(U) := C^0(U; \mathbb{R}) \xrightarrow{\delta} C^1(U; \mathbb{R}) \xrightarrow{\delta} C^2(U; \mathbb{R}) \rightarrow \dots$$

are both sheaves (of chain complexes) plus

$$\int : \Omega_M^i \rightarrow C_{0,1,2}^i \text{ is a map of sheaves.}$$

$\Rightarrow$  Can check locally, which is implied by  
Poincaré lemmas.

• What is microlocal .

.. Local: We say  $F$  is trivial on  $U$  if  $F|_U = 0$ .

Observation:  $\mathcal{U} = \{U_\alpha\}$  is a family s.t.  $F|_{U_\alpha} = 0$

for all  $\alpha$ , then  $F|_U = 0$  where  $U := \bigcup_\alpha U_\alpha$

Proof: For any  $V \subseteq U$ ,  $\{V \cap U_\alpha\}$  is a cover of  $V$  so

$$\Gamma(V; F) \hookrightarrow \prod_\alpha \Gamma(V \cap U_\alpha; F) = 0$$

Conclusion: There is a largest  $U$  s.t.  $F|_U = 0$ .

Def<sup>n</sup>: The support of  $F$  is  $\text{supp}(F) := U^c$ .

In particular,  $\text{supp}(F)$  is a closed set.

Example: Consider  $X = \mathbb{R}$  and  $F$  given by

$$F(U) = \begin{cases} 0, & \text{if } 0 \notin U \\ \mathbb{Z}, & \text{if } 0 \in U. \end{cases}$$

$$\text{If } 0 \in V \subseteq U, \text{ then } F(U) \rightarrow F(V) \\ \mathbb{Z} = \mathbb{Z}$$

and 0 otherwise.

On picture,



$\bar{F} = \mathbb{Z}_{\{0\}}$  the skyscraper sheaf at 0.

$$\text{Supp}(\mathbb{Z}_{\{0\}}) = 0$$

Similarly, one can consider  $\mathbb{Z}_{[0, \infty)}$  the locally constant sheaf supported on  $[0, \infty)$ .

$$0 \leftarrow \mathbb{Z} = \mathbb{Z}$$

Assume  $U$  are open intervals, then

$$\mathbb{Z}_{[0, \infty)}(U) = \begin{cases} 0, & \text{if } U \cap [0, \infty) = \emptyset \\ \mathbb{Z}, & \text{if } U \cap [0, \infty) \neq \emptyset. \end{cases}$$

A variant is  $\mathbb{Z}_{(0, \infty)}$ , which on picture is

$$0 = 0 \longrightarrow \mathbb{Z}$$

Note:  $\text{supp}(\mathbb{Z}_{[0, \infty)}) = \text{supp}(\mathbb{Z}_{(0, \infty)}) = [0, \infty)$

but the (co)direction of change is different.

Let  $M^n$  be a  $C^1$ -manifold and  $F$  a sheaf.

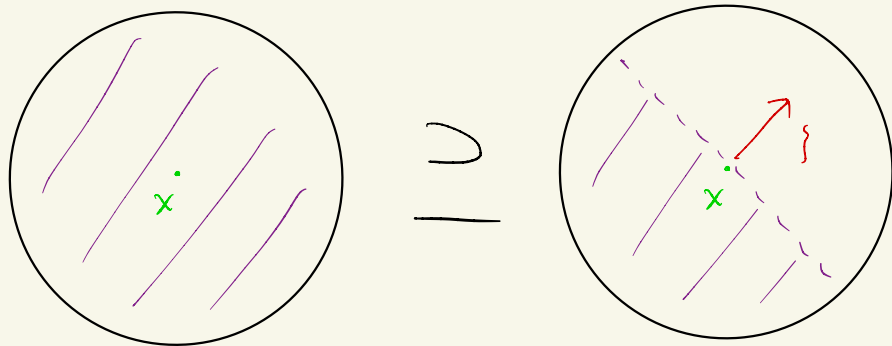
Want: An invariant  $SS(F) \subseteq T^*M$  that encode the "nontrivial" codirections,

Approximate definition:  $SS(F) \cap \mathcal{Q}_m = \text{supp}(F)$ .

For  $(x, \xi) \in T^*M$ , pick a chart  $U \cong B^n$ , then

$(x, \xi) \notin SS(F)$  if

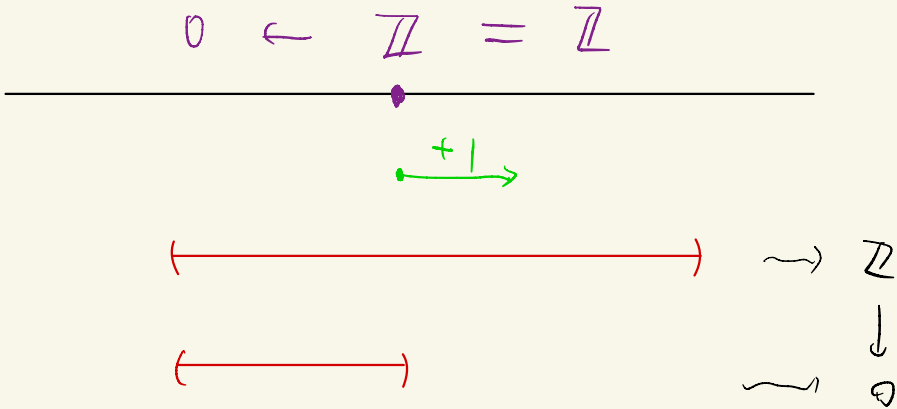
$$\Gamma(B(x); F) \xrightarrow{\cong} \Gamma(B(x) \cap \{\langle \cdot, \xi \rangle < 0\}; F)$$



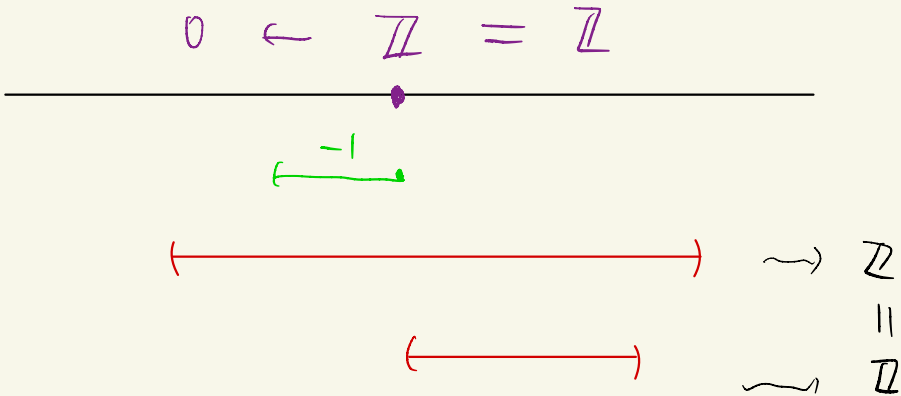
In other words,  $(x, \xi) \notin SS(F)$  if the sections don't propagate uniquely. Note that  $SS(F)$  is conic by definition.

Back to the example.

For  $\mathbb{Z}_{[0, \infty)}$

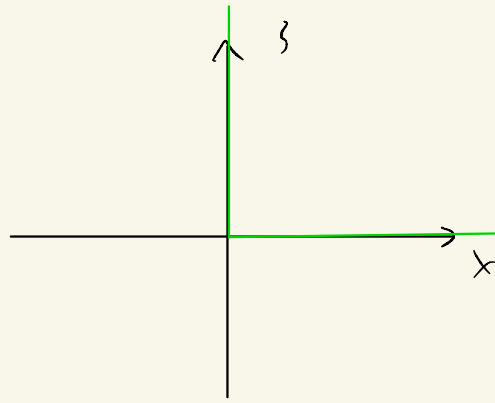


$\Rightarrow (0, 1) \notin \text{SS}(\mathbb{Z}_{[0, \infty)})$

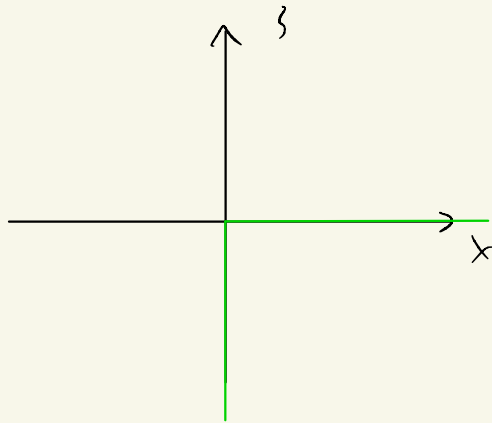


$\Rightarrow (0, -1) \notin \text{SS}(\mathbb{Z}_{[0, \infty)})$

$$\Rightarrow SS(\mathbb{Z}_{[0, \infty)}) =$$



$$\text{Similarly, } SS(\mathbb{Z}_{(0, \infty)}) =$$



With this definition, it's natural to localize on the cotangent bundle, hence the term microlocal.

Precisely, for  $\Omega \subseteq T^*M$ , one considers

$$\text{Sh}(M) / \text{Sh}_{\Omega^c}(M) \quad \text{where} \quad \text{Sh}_X(M) := \{F \mid \text{SS}(F) \subseteq X\}$$

Note " $\Omega_0 \subseteq \Omega$ "  $\Leftrightarrow$  " $\Omega^c \subseteq \Omega_0^c$ "

$$\Rightarrow \text{Sh}_{\Omega^c}(M) \subseteq \text{Sh}_{\Omega_0^c}(M)$$

$$\rightsquigarrow \text{Sh}(M) / \text{Sh}_{\Omega^c}(M) \longrightarrow \text{Sh}(M) / \text{Sh}_{\Omega_0^c}(M)$$

builds you a sheaf of categories  $\text{msh}_{T^*M}$ .

• Connection to symplectic geometry:

Recall  $T^*M$  has a natural symplectic structure coming from a Liouville form  $\alpha_M$ .

A diffeomorphism  $\chi: U \xrightarrow{\cong} V$ ,  $U \subseteq T^*M$  and  $V \subseteq T^*N$ , is a contactomorphism  $\mathbb{M}$   $\chi^*(\alpha_N) = \alpha_M$

Fact (Kashiwara-Schapira):

①  $SS(F)$  is always coisotropic and, under mild regularity assumptions,  $SS(F)$  is a Lagrangian  $\mathbb{M}$   $SS(F)$  is constructible.

②  $\text{msh}$  is a contact invariant: For  $U, V$  small,

$\chi: U \xrightarrow{\cong} V$  induces an isom.

$$\chi^*(\text{msh}_{T^*N}|_V) \cong \text{msh}_{T^*M}|_U.$$

Recent development:

Thom (Guillemin-Kashimura-Schubert)

$\Downarrow$   $\Phi: [0,1] \times T^*M \rightarrow T^*M$  is a contact isotopy,

then  $\chi := \Phi(-,1)$  induces an auto-equivalence on  $\text{Sh}(M)$ .

Construction: (Shende, Nadler-Shende).

Let  $W$  be an exact symplectic manifold.

Fix a Lagrangian subbundle  $\phi \subseteq TW$ . Then

$\exists$  a sheaf of categories  $\text{msh}_{W,\phi}$  locally

given by  $\text{msh}_{T^*M}$  and recovers the earlier notion

when  $W = T^*M$  w.  $\phi = \phi_M: (x,1) \mapsto T_x^*M \subseteq T_{(x,1)}(T^*M)$

( $\phi$  is called a polarisation.)

Some applications :

① Symplectic side : Sheaves model Fukaya categories,

Thm (Ganatra - Pardon - Shende) : Let  $\Lambda \subseteq T^*M$  be Lagrangian.

Then  $\text{Ind}[\mathcal{W}(T^*M; \Lambda)] \simeq \text{Sh}_\Lambda(M)$ .

In fact, for a polarized Weinstein manifold  $X$ , we have

$$\mathcal{W}(X) \simeq \text{msht}_{\Gamma_X}(\Gamma_X)$$

Other applications : Non-displaceability (after Tamarkin),

$\mathbb{C}^n$ -geometry etc...

② Representation theory side:

Thm (Beilinson - Bernstein)  $G$ : reductive group /  $\mathbb{C}$

$$(\mathcal{U}(\mathfrak{g}) / \ker \chi) - \text{Mod} \cong \mathcal{D}_{G/\mathbb{P}} - \text{Mod}$$

where  $\chi: \mathcal{Z}(\mathcal{U}(\mathfrak{g})) \rightarrow \mathbb{C}$  is a certain character.

Thm (The Riemann-Hilbert correspondence, Kashiwara (Kawai))

$X$ : complex manifold. Then,

$$\mathcal{D}_X - \text{Mod}_{\text{rh}} \cong \text{Perv}(X).$$

→ Proof of Kazhdan-Lusztig conjecture

by Brylinski & Kashiwara.

(Beilinson-Bernstein has an independent proof)  
around the same time.

Other research directions:

- ① Mirror symmetry
- ② non-displaceability & non-squeezing invariants
- ③  $C^0$ -symplectic / contact geometry
- ④ Noncommutative geometric str. on Fukaya categories