Superpotentials in Mirror Symmetry

Christopher Kuo

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Abstract

In this talk, we discuss the appearance of superpotentials for matrix factorizations from the symplectic side of Mirror Symmetry in the SYZ construction, with the monomial terms corresponding to components of the canonical boundary divisor. We follows Auroux's survey papers [Aur07] and [Aur09b]. Some basics note can be found in his course website [Aur09a]. Any errors introduced are mine.

1   Introduction

Mirror symmetry refers to a set of statements about pairs of manifolds X, $X^\vee$ called mirror pairs. Morally, going to the mirror should swap the symplectic and complex data. There are two main formulations of it. One is the Homological mirror symmetry proposed by Kontsevich which states that for a symplectic manifold X, there is a complex manifold $X^\vee$ such that $\mathrm{Fuk}(X) \cong \mathrm{Coh}(X^\vee)$. The other is the Strominger-Yau-Zaslow (SYZ) conjecture which provides a way to construct mirror pairs through torus fibration.

2   Calabi-Yau cases and the SYZ conjecture

Recall a Calabi-Yau manifold is a compact Kähler manifold X with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$. In this talk, we call a quadruple (X, ω, J, Ω) an almost Calabi-Yau structure if (X, ω, J) is the underlining Kähler manifold and $\Omega \in \Omega^{(n,0)}$ is a nonvanishing holomorphic volume form. A cartoon version of the SYZ conjecture is the following.

Conjecture 2.1 (Strominger, Yau, and Zaslow)**.** *Let X be a n-dimensional Calabi-Yau manifold. There exists a special Lagrangian torus fibration $f : X \to B$ and the mirror manifold $X^\vee$ is the total space of the dual torus fibration.*

Recall that a Lagrangian $L \subseteq X$ is a special Lagrangian if $\arg(\Omega\big|_L)$ is constant. For simplicity, we assume $\mathrm{Im}(\Omega\big|_L) = 0$ by multiplying a factor. Assume such fibration f exists, the base space B can be seen as moduli space of Lagrangian fibers. Thus, the tangent space at $b \in B$ governs infinitesimal deformation of the special Lagrangian $L = f^{-1}(b)$.

Proposition 2.2. *Infinitesimal deformation of special Lagrangian is classified by $H^1(L; R)$.*

More precisely, there exists isomorphisms $NL \cong T^*L \cong \Lambda^{n-1}T^*L$ where the correspondences are given by
$$v \longleftrightarrow -\iota_v\omega \longleftrightarrow \iota_v \mathrm{Im}\,\Omega.$$

Let v be a normal vector field. We can construct a deformation by $L_t = j_t(L)$ where $j_t(p) = \exp_p(tv(p))$. This is an infinitesimal deformation of special Lagrangians if and only if $\alpha := -\iota_v \omega$ satisfies $d\alpha = 0$ and $d^*(\psi\alpha) = 0$ where ψ is defined by $\iota_v \operatorname{Im}\Omega = \psi *_g \alpha$.

Definition 2.3. We call such α a ψ-harmonic 1-form and denote the space consisting such ψ-harmonic 1-from $\mathcal{H}_\psi^1(L)$. Standard Hodge theory argument shows that $\mathcal{H}_\psi^1(L) \cong H^1(L;\mathbb{R})$.

To get a complex n-fold, we complexify B by considering flat $U(1)$-connections. Let M the moduli space of special Lagrangian fibers equipped with a flat $U(1)$-connection on the trivial bundle, i.e, points in M are given by pairs (L, ∇) up to gauge equivalence. The fibration $f^\vee : M \to B$ defined by $(L, \nabla) \mapsto L$ is dual to f.

Proposition 2.4. *The tangent space at (L, ∇) can be identified with $\mathcal{H}_\psi^1(L) \otimes \mathbb{C}$ by*

$$(v, \alpha) \mapsto -\iota_v \omega + i\alpha$$

where $v \in C^\infty(NL)$ is an infinitesimal deformation and $\alpha \in \Omega^1(L, \mathbb{R})$ is a ψ-Harmonic one form. There is an (almost) Calabi-Yau structure $(M, \omega^\vee, J^\vee, \Omega^\vee)$ given by

$$\Omega^\vee\left((v_1, \alpha_1), \cdots, (v_n, \alpha_n)\right) = \int_L (-\iota_{v_1}\omega + i\alpha_1) \wedge \cdots (-\iota_{v_n}\omega + i\alpha_n),$$

$$J^\vee(v, \alpha) = (a, -\iota_v \omega),$$

here a is defined by $\iota_a \omega = \alpha$ and

$$\omega^\vee\left((v_1, \alpha_1), (v_2, \alpha_2)\right) = \int_L \alpha_2 \wedge \iota_{v_1} \operatorname{Im}\Omega - \alpha_1 \wedge \iota_{v_2} \operatorname{Im}\Omega.$$

(The formula for $\omega^\vee$ assumes that $\int_L \operatorname{Re}\Omega$ has been suitably normalized.)

We can find a complex coordinate for M. For $A \in H_2(X, L; \mathbb{Z})$ with $\partial A \neq 0$ in $H_1(L; \mathbb{Z})$, define

$$z_A = e^{-\int_A \omega} \operatorname{hol}_\nabla(\partial A) : M \to \mathbb{C}^*.$$

This function is a locally well-defined. If one can find $A_i \in H_2(X; L, \mathbb{Z}), i = 1, \cdots, n$ such that ∂A_i forms a basis of $H_1(L; \mathbb{Z})$, then z_{A_i} gives a complex local coordinate of M. Otherwise, one needs to replace the definition of z_A by using a reference curve.

Example 2.5. Let $X = \mathbb{C}/(\mathbb{Z} + i\rho\mathbb{Z})$ the a two torus. Equip it with with the symplectic form $\omega = \frac{\lambda}{\rho} dx \wedge dy$, a holomorphic volume form $\Omega = dz$ and an S^1-fibration $\pi : T^2 \to S^1$ defined by $\pi(x, y) = y$. The fibers are of the form $L_t = \{y = t\}$. Choose the circle L_0 as the reference curve, and the holomorphic coordinate is $z^\vee(L_y, d + i\theta dx) = e^{-2\pi \frac{\lambda}{\rho} y} e^{i\theta}$. Taking $\frac{1}{2\pi i} \log$, we see that the mirror is another two torus $X^\vee = \mathbb{C}/(\mathbb{Z} + i\lambda\mathbb{Z})$ with $\Omega^\vee = dz^\vee$ and $\omega^\vee = \frac{1}{2\pi} d\theta \wedge dy$.

In real life, most Calabi-Yau manifolds don't have any genuine torus fibration and singular fibers can exist. A natural idea is to to consider the above construction away from singularities and then extend it. Modification needs to be done.

3 Beyond Calabi-Yau and the superpotential

There are also statements for mirror symmetry in more general situations. Assume (X, ω, J) is a compact Kähler manifold of complex dimension n, and $D \subseteq X$ is an effective divisor representing the anticanonical class, with at most normal crossing singularities. The defining section σ of D induces a nonvanishing holomorphic n-form $\Omega = \sigma^{-1}$ with simple poles along D.

We want to use a similar construction to find a mirror for $X \setminus D$, i.e., by considering a moduli of some suitable special Lagrangian tori. The open manifold $X \setminus D$ contains essentially the same Lagrangian as X. But special Lagrangian tori tend to bound families of holomorphic disks in X and the Floer homology will be obstructed.

Definition 3.1. For $\beta \in \pi_2(X, L)$, denote $\mathcal{M}_k(L, \beta)$ the moduli space of holomorphic disks in (X, L) with k boundary marked points, and $\overline{\mathcal{M}}_k(L, \beta)$ the compactified moduli space. The expected dimension of $\mathcal{M}_k(L, \beta)$ is $n - 3 + k + \mu(\beta)$ where $\mu(\beta)$ is the Maslov index. (See [Aur13] for the detailed definition.) In the case when $L \subseteq X \setminus D$ is a special Lagrangian, $\mu(\beta)$ is given by twice the intersection number $\beta \cdot [D]$.

Denote $\mathcal{L} = (L, \nabla)$. There is an element $m_0(\mathcal{L}) \in CF^*(\mathcal{L}, \mathcal{L})$ which is defined by

$$m_0(\mathcal{L}) = \sum_{\beta, \mu(\beta)=2} z_\beta(\mathcal{L}) ev_*[\overline{\mathcal{M}}_1(L, \beta)]^{vir}.$$

Here we use the singular chain model for the Floer homology and $ev : \overline{\mathcal{M}}_k(L, \beta) \to L$ is the evaluation map sending $\mu \in \overline{\mathcal{M}}_k(L, \beta)$ to the marked point. (See [FOOO06])

The Hom complex $CF^*(\mathcal{L}, \mathcal{L}')$ has left $CF^*(\mathcal{L}, \mathcal{L})$-module and right $CF^*(\mathcal{L}', \mathcal{L}')$-module structures both denoted by m_2. The geometry of the boundary of 1-dimensional moduli space implies that the operation m_2 on $CF^*(\mathcal{L}, \mathcal{L}')$, satisfies,

$$m_1^2 = m_2\left(m_0(\mathcal{L}'), \cdot\right) - m_2\left(\cdot, m_2(\mathcal{L})\right).$$

Consider now only weakly unobstructed Lagrangians $\mathcal{L}$, i.e., $m_0(\mathcal{L}) = \lambda 1_\mathcal{L}$ in $C^*(\mathcal{L}, \mathcal{L})$. One can see that, Floer homology HF^* is defined only for Lagrangians with the same m_0. A fact to be noted is that $HF^*(\mathcal{L}, \mathcal{L})$ is generically trivial due to contributions of holomorphic discs in (X, L) to the Floer differential.

On the mirror side (when $X^\vee$ is affine), a similar structure comes from matrix factorization $MF(\check{X}, W)$. The objects are given by $\mathbb{Z}/2$-graded projective $\mathbb{C}[X^\vee]$-module with odd endomorphism $\delta^2 = (W - \lambda)\,\mathrm{id}$. The differentail on $\mathrm{Hom}\left((P_1, \delta_1), (P_2, \delta_2)\right)$ squares to $(\lambda_1 - \lambda_2)\,\mathrm{id}$. For a fixed λ, the corresponding objects forms a category which is isomorphic to $D^b_{sing}(W, \lambda) = \mathrm{Coh}(W^{-1}(\lambda))/\mathrm{Perf}(W^{-1}(\lambda))$ and is non-trivial only when $W^{-1}(\lambda)$ is a singular fiber. (See[Orl04])

This motivates the following definition. Recall in the almost Calabi-Yau case, z_β is a complex function on the moduli space M.

Conjecture 3.2. *The mirror of X is the Laudau-Giznburg model $(X^\vee, W)$ where*

(a) $X^\vee$ is a mirror of the almost Calabi-Yau manifold $X \setminus D$, i.e., a (corrected and completed) moduli space of special Lagrangian tori in $X \setminus D$ equipped with rank 1 local system;

(b) $W : X^\vee \to \mathbb{C}$ *is a holomorphic function defined by*

$$W(p) = \sum_{\beta,\mu(\beta)=2} n_\beta(\mathcal{L}_p) z_\beta(\mathcal{L}_p)$$

where $n_\beta(\mathcal{L}_p) = \deg\left(ev_*[\overline{\mathcal{M}}_1(L, \beta)]^{vir}\right)$ which is the (virtual) number of holomorphic discs in the class β passing through a generic point of L.

There are issues with this conjecture. One of them is that the number $n_\beta(\mathcal{L}_p)$ might not be constant and one needs to correct this formula by adding extra instanton correction terms.

An example for which conjecture actually holds is when (X, ω, J) is a smooth toric Fano variety, i.e., its anticanonical divisor is ample. Recall that for any toric variety, there is a moment map $\phi : X \to \mathbb{R}^n$ whose image is a polytope Δ and whose fibers are special Lagrangian T^n-orbits. The pre-image of the interior $\phi^{-1}(\Delta)$ is a dense open complex torus $(\mathbb{C}^*)^n \subseteq X$, whose complement is the anticanonical divisor D, sometimes called the toric boundary. On $(\mathbb{C}^*)^n$ pick the standard holomorphic volume form $\Omega = d\log x_1 \wedge \cdots d\log x_n$.
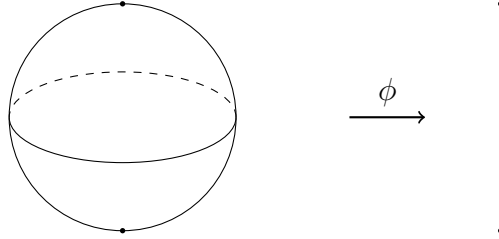
In this case, chose M to be the moduli of form (L, ∇) where L is a T^n - orbit and let

$$z_j(L, \nabla) = e^{-2\pi\phi_j(L)} \operatorname{hol}_\nabla(\gamma).$$

Under the identification $X \setminus D \cong (\mathbb{C}^*)^n$, L is of the form $S^1(r_1) \times \cdots \times S^1(r_n) \subseteq (\mathbb{C}^*)^n$, ϕ_j is the j-th component of ϕ and $\gamma_j := [S^1(r_j)]$. One can check that M is biholomorphic to a bounded open subset of $(\mathbb{C}^*)^n$.

Proposition 3.3. *Under the identification $M \hookrightarrow (\mathbb{C}^*)^n$, the superpotential is given by the Laurent polynomial*

$$W = m_0(L, \nabla) = \sum_{F \ facet} e^{-2\pi(\alpha F)} z^{\nu(F)}.$$

Here $\nu(F)$ is the inward primitive integer normal vector of F and $\alpha(F)$ is the constant in the defining equation $\langle \nu(F), \phi \rangle + \alpha(F) = 0$. This small mirror can be completed to the whole $(\mathbb{C}^*)^n$ with superpotential given by the same formula.

Remark 3.4. The positivity assumption of being Fano ensures compactness of moduli spaces of Maslov index 2 holomorphic discs in (X, L) and that certain bubbling phenomenon won't show up.

4 Wall-crossing

Now let's look at an example where the wall-crossing phenomenon shows up. Consider the function $f : \mathbb{C}^2 \to \mathbb{C}$ defined by $f(x, y) = xy$. Set $X = \mathbb{C}^2$ with the standard Kähler structure, $D = f^{-1}(\epsilon)$ and $\Omega = dx \wedge dy/(xy - \epsilon)$.

The special Lagrangian tori we will consider in this case are those of the form

$$T_{r,\lambda} = \{(x, y) \in \mathbb{C}^2 \big| |xy - \epsilon| = r, \mu_{S^1}(x, y) = \lambda\}, (r, \lambda) \neq (|\epsilon|, 0)$$

where $\mu_{S^1}(x, y) = \frac{1}{2}(|x|^2 - |y|^2)$, a moment map for the symplectic form $\omega = \frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y})$ and S^1 action $e^{i\theta}(x, y) = (e^{i\theta}x, e^{-i\theta}y)$. When $(r, \lambda) = (|\epsilon|, 0)$, $T_{r,\lambda}$ has a nodal singularity at the origin.

We chose M to be the moduli space of pairs $(T_{r,\lambda}, \nabla)$ for some $(r, \lambda) \neq (|\epsilon|, 0)$ and it admits a complex structure given by the function $z_\beta = e^{-\int_\beta \omega} \operatorname{hol}_\nabla(\partial\beta)$.

Now we want to find a coordinate system for M and the superpotential W under this coordinate. The first place we need to look at is when $r = |\epsilon|$. When $\lambda > 0$, the tori $T_{|\epsilon|,\lambda}$ intersect the x-axis in a circle which bounds a class of Maslov index 0 disc α. When $\lambda < 0$, the intersection with the y-axis bounds $-\alpha$.

Definition 4.1. A torus $T_{r,\lambda}$ is Clifford type if they can be deformed to $S^1(r_1) \times S^2(r_2)$ and is of Chekanov type which can be deformed to $|xy - \epsilon| = r$, $|x| = |y|$.

To find the Maslov index 2 discs, assume $r > |\epsilon|$ first. The tori $T_{r,\lambda}$ are of and there are two class of Maslov index 2 discs β_1 and β_2 corresponding to $D^2(r_1) \times \{y\}$ and $\{x\} \times D^2(r_2)$. Denote $z_i = z_{\beta_i}$ for $i = 1, 2$. They satisfies $z_1/z_2 = z_\alpha = w$ and the superpotential is given by $W = z_1 + z_2$ in this case. For $r < |\epsilon|$, the tori $T_{r,\lambda}$ are of Chekanov type which can be deformed to and there is only one class of Maslov index 2 disc β_0 given by $y = ax$, $|a| = 1$. Denote $\beta_0 = u$. The superpotential is given by $W = u$ in this case.

When increasing r pass $r = |\epsilon|$, for $\lambda > 0$, the class β_0 deforms into β_2 which suggests us to use the gluing $u = z_2$, $w = z_1/z_2$. However, similar reasoning suggests us to use $u = z_1$, $w = z_1/z_2$ which doesn't match. This is because there's monodromy around the singular fiber $T_{|\epsilon|,0}$ so z_i's don't extend globally although w does.

Consider $\lambda > 0$. The way to fix is to add some instanton correction terms. When increasing r pass $r = |\epsilon|$, in addition to β_2, β_0 also give rise to new class $\beta_2 + \alpha = \beta_1$ by attaching the exceptional disc bounded by $T_{|\epsilon|,\lambda}$. So we use $u = z_2 + z_1 = z_2(1 + w)$ instead. Similarly, we use $u = z_1(1 + w^{-1})$ for $\lambda < 0$. This way the gluing matches and the superpotential is given by $W = u = z_1 + z_2$.

Proposition 4.2. *After completion, the mirror $X^\vee$ is given by*

$$X^\vee = \{(u, v, w) \in \mathbb{C}^2 \times \mathbb{C}^* \big| uv = 1 + w\}, W = u.$$

Remark 4.3. If we started with $\mathbb{CP}^2$ and delete the $\mathbb{CP}^1$ at infinite along with D, we would get the same $X^\vee$ with

$$W = z_1 + z_2 + \frac{e^{-\int_{\mathbb{CP}^1} \omega}}{z_1 z_2}$$

since $T_{r,\lambda}$ would bound more additional Maslov index 2 discs.

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