Introduction to Microlocal Sheaf Theory [V5D4]

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1 Introduction

Microlocal geometry originated in the study of partial differential equations. This development began with Sato's specialization and microlocalization functors [Sat69] and was followed shortly thereafter by Hörmander's wave front set [Hor73]. These tools allow one to analyze not only the singularities of solutions (in this context, the non-propagation of solutions) but also the codirections of those singularities. By abstracting this approach, Kashiwara and Schapira introduced the concept of microsupport [KS85], which is used to detect various properties of sheaves; together with microlocalization, these form the core tools of microlocal sheaf theory.

A major advancement occurred in the late 2000s with the discovery of deep applications to symplectic geometry. It had long been known that microlocal sheaf theory is closely connected to the subject; for instance, the microsupport of any sheaf is always coisotropic. However, it was only in 2006 that Nadler and Zaslow [NZ09, Nad09] demonstrated that sheaves can be used to compute objects in a version of the Fukaya category of the cotangent bundle. Around the same time, in 2008, Tamarkin [Tam18] proved non-displaceability results using microlocal sheaf theory; systematic results of this kind had previously been achievable only via Floer theory. Since then, numerous further results intertwining the two fields, including applications to mirror symmetry, have been obtained, and the area continues to grow rapidly.

2 Description of the course

The goal of this course is twofold. On the one hand, we will review the construction of the theory as developed in Kashiwara and Schapira's book *Sheaves on Manifolds* [KS94], along with its further development using symplectic geometric methods. Examples include the method of isotopies of sheaves [GKS12] and the notion of microsheaves on an exact symplectic manifold [NS21]. On the other hand, we will provide overviews of its applications depending on the interests of the audience. To name a few: microsheaves model Fukaya categories [GPS20] and, as a result, provide a clean proof for mirror symmetry [Kuw17]. Another direction is the study of representation theory [HTT08] through the Riemann–Hilbert correspondence.

3 Prerequisites and suggested readings

Knowledge of point-set topology and manifold theory will be assumed; experience with homological algebra and six-functor formalisms of any kind will be helpful. In general, I will try to recall the necessary material as we proceed, but familiarity with the following materials will certainly be handy:

1. The higher category theory developed by Lurie: [Lur09, Lur17, Lur18]. The part most relevant to us is [Lur17, Chapter 1], where the theory of stable ∞-categories (an enhancement of triangulated categories) is introduced.

2. The six-functor formalism within the higher categorical framework. A very general theory is explained by Scholze in [Sch25], but for a more topologically minded audience, Volpe's thesis [Vol21] will suffice.

Please feel free to reach out if you have a particular direction of applications in mind.

References

[GKS12] Stéphane Guillermou, Masaki Kashiwara, and Pierre Schapira. Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems. *Duke Math. J.*, 161(2):201–245, 2012.

[GPS20] Sheel Ganatra, John Pardon, and Vivek Shende. Microlocal Morse theory of wrapped Fukaya categories. *arXiv:1809.08807v2*, 2020.

[Hor73] Lars Hormander. *An introduction to complex analysis in several variables.* Elsevier, 1973.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*, volume 236 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 2008. Translated from the 1995 Japanese edition by Takeuchi.

[KS85] Masaki Kashiwara and Pierre Schapira. *Microlocal study of sheaves*, volume 128 of *Astérisque.* Société Mathématique de France (SMF), Paris, 1985.

[KS94] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1994. With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original.

[Kuw17] Tatsuki Kuwagaki. The nonequivariant coherent-constructible correspondence for toric stacks. *arXiv:1610.03214*, 2017.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2009.

[Lur17] Jacob Lurie. Higher algebra. *Preprint, available at https://www.math.ias.edu/ lurie/*, 2017.

[Lur18] Jacob Lurie. Spectral algebraic geometry. *Preprint, available at https://www.math.ias.edu/ lurie/*, 2018.

[Nad09] David Nadler. Microlocal branes are constructible sheaves. *arXiv:math/0612399v4*, 2009.

[NS21] David Nadler and Vivek Shende. Sheaf quantization in Weinstein symplectic manifolds. *arXiv:2007.10154v2*, 2021.

[NZ09] David Nadler and Eric Zaslow. Constructible sheaves and the Fukaya category. *J. Amer. Math. Soc.*, 22(1):233–286, 2009.

[Sat69] Mikio Sato. Hyperfunctions and partial differential equations. In *Proc. int. conf. on functional analysis, tokyo*, volume 9, 1969.

[Sch25] Peter Scholze. Six-functor formalisms. *Lecture Notes*, 2025.

[Tam18] Dmitry Tamarkin. Microlocal condition for non-displaceability. In *Algebraic and analytic microlocal analysis. AAMA, Evanston, Illinois, USA, May 14–26, 2012 and May 20–24, 2013. Contributions of the workshops*, pages 99–223. Cham: Springer, 2018.

[Vol21] Marco Volpe. The six operations in topology. Preprint, arXiv:2110.10212 [math.AT] (2021), 2021.