INTRODUCTION TO MICROLOCAL SHEAF THEORY

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Abstract.

Contents

1. Introduction

1.1. **From sheaves to microlocal geometry.** Sheaf theory originated as a natural extension of algebraic topology; as Houzel [23] recounts, Jean Leray developed the theory's foundational concepts while teaching an algebraic topology course at the University of Captivity at Oflag XVII-A in Austria. The question he was then considering concerned the existence of global solutions to nonlinear equations in fluid dynamics; he needed a general framework to bridge the gap between local analysis and global geometry. This procedure of patching local solutions into a global one is now fundamental in modern mathematics and is often referred to as the "local-to-global" principle.

In short, a sheaf on a space X is an assignment F that sends each open set U to a set, group, or ring of sections $F(U)$. Furthermore, for an inclusion $V \subseteq U$, there must be a natural restriction map $F(U) \to F(V)$, and these maps must respect the topology of X in the same way that functions do: two sections s_1 and s_2 are equal if they agree on an open cover. Moreover, a family of sections $\{s_\alpha\}$, each defined on a member of an open cover $\{U_\alpha\}$, glues to a global section on X provided that, for each pair of indices α and β, the restrictions $s_\alpha|_{U_\alpha \cap U_\beta}$ and $s_\beta|_{U_\alpha \cap U_\beta}$ agree. As one can see, real-valued functions, with possibly varying regularity requirements such as continuity or differentiability, and solutions to a given PDE are all natural examples of sheaves.

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Progress, however, did not stop here. Like functions, sheaves can be viewed as geometric objects in their own right, and in many ways they naturally categorify functions. For example, one can pull back functions along maps and, if the functions take values in numbers, one can also multiply them. By categorifying these operations, one can pull back sheaves and, if the sheaves take values in vector spaces, one can also tensor them. Categorification also yields additional structure: there are adjoint notions of pushforward and, in the vector-space-valued case, of internal Hom. In fact, there are two further "exotic" operations that bring the total to six.

The six-functor formalism not only allows us to integrate classical algebraic topology results into a single framework, but it also provides a clean foundation for studying the microlocal aspects of the theory and connecting them with symplectic geometry: over a manifold M, sheaves as geometric objects naturally live not on the base but on the cotangent bundle T^*M. This is realized through the notion of microsupport. For a given sheaf F on M, one can associate a subset $\mathrm{SS}(F) \subseteq T^*M$, the microsupport of F, which records the codirections along which sections do not propagate. This can be further used to construct a sheaf $\mu\mathrm{sh}_{T^*M}$, valued in categories, of microsheaves; in principle, this assigns to an open set $\Omega \subseteq T^*M$ the category of sheaves, quotiented by those microsupported outside Ω. Even more remarkably, this latter invariant can be globalized using symplectic geometry: for an exact symplectic manifold X, up to the vanishing of a certain obstruction, there exists a sheaf $\mu\mathrm{sh}_X$ locally glued out of $\mu\mathrm{sh}_{T^*M}$, which is known to contain the wrapped Fukaya category of X.

1.2. **The course and its goals.** The goal of this course is to give an overview of the current state of the field of microlocal sheaf theory. Our discussion will be divided into two different but intertwining concerns. The first regards the foundations. Kashiwara and Schapira's tome *Sheaves on Manifolds* [26] encompassed most of the knowledge regarding the subject at the time of its publication and has since served as the standard textbook.

More than thirty years have passed since then. While the book still maintains its prominence both as a textbook and as a source of research directions, many advances in related fields have taken place. First, the higher categorical foundations, developed by many and later consolidated in Lurie's three books [34, 35, 36] together with his website [37], have transformed the landscape of category theory and homological algebra, which form the foundation upon which microlocal sheaf theory rests. A first goal is therefore to study Chapters 1–3 of [26] in this framework, i.e., to construct the six-functor formalism in topology within this modern setting. We will follow [52], where the material is covered and generalized; for example, many of the restrictions on coefficients and the boundedness assumptions are lifted. Furthermore, a notion of abstract six-functor formalism has been recently established [38, 21, 45], subsuming the one in [26] as a special case.

Our second goal is to discuss the microlocal aspects of the theory, roughly Chapters 4–8 of [26]. The central objects of study here are the microsupport $\mathrm{SS}(F)$, which has acquired a new interpretation through the Ω-lens [20] in the past decade, and the notion of microsheaves $\mu\mathrm{sh}$. Building on these two notions, many of the symplectic aspects of the theory were discovered after the publication of [26]. For example, Guillermou, Kashiwara, and Schapira construct an action of contact isotopies not only on microsheaves but on the categories of sheaves themselves [19], further justifying the intuition of treating constructible sheaves as Lagrangians. Additionally, we will discuss the construction of Nadler and Shende [46, 40], which produces a sheaf of categories $\mu\mathrm{sh}_{X;\tau}$ for any exact symplectic manifold X equipped with a Maslov datum τ.

The second part of the course will concern applications. Since microlocal sheaf theory has applications to very different subjects, the exact content will depend on the audience. Here we mention two main applications of the theory.

The first field of application is symplectic geometry. A central question in symplectic geometry is to construct algebraic invariants for Lagrangians; these are often referred to as their quantizations. The classical approach is through Floer theory, a version of infinite-dimensional Morse cohomology. To obtain a more powerful invariant, given a symplectic manifold X, one often collects an admissible family of Lagrangians and forms a Fukaya category $\mathrm{Fuk}(X)$ whose objects are such Lagrangians and whose morphisms, for $L, K \in \mathrm{Fuk}(X)$, are given by the Floer homology $\mathrm{HF}^*(L, K)$.

An alternative approach to quantizing Lagrangians originates with Nadler and Zaslow in [41, 39]. There, they define the infinitesimal Fukaya category $\mathrm{Fuk}_\epsilon(T^*M)$ for a manifold M and show that $\mathrm{Fuk}_\epsilon(T^*M) \simeq \mathrm{Sh}_{\mathbb{R}\text{-}c}(M)$, the category of constructible sheaves. In other words, the Lagrangians they consider admit sheaf quantizations. Related ideas were also pursued by Viterbo [51] and Guillermou [18], as well as by Ganatra, Pardon, and Shende [16, 15, 17], whose work led to the proof of Kontsevich's conjecture [28].

Another, more classical application is to geometric representation theory. This begins with Beilinson and Bernstein's localization theorem [6], which identifies $\mathfrak{g}$-representations—where $\mathfrak{g}$ is the Lie algebra of a complex reductive group G—with D-modules on the flag variety G/B. In the same paper, they also give a proof of the Kazhdan–Lusztig conjecture. One can then further pass from D-modules to perverse sheaves via the Riemann–Hilbert correspondence [24, 25]; exploiting the simpler structure of perverse sheaves, an independent proof of the conjecture was given by Brylinski and Kashiwara in the same year [9]. A textbook account can be found in [22].

There is a saying, usually attributed to Okounkov [42], that "symplectic resolutions are the Lie algebras of the 21st century." Indeed, a central notion in geometric representation theory is the category $\mathcal{O}$, and Webster et al. have defined such a notion for symplectic resolutions in [8, 7], using a deformation-theoretic version of D-modules. Building on the work of Andronikof [1] and Waschkies [53], and in joint work with Côté at the University of Bonn, Nadler, and Shende, the author was able to microlocalize the Riemann–Hilbert correspondence [10] and provide a description of the category $\mathcal{O}$ of symplectic resolutions in terms of perverse microsheaves.

1.3. **Other notable directions.** In this section, we list a few other applications of microlocal sheaf theory for the interested reader.[1]

(a) Around the same time as Nadler and Zaslow, Tamarkin in [48][2] proves a result concerning non-displaceability: the question of whether two Lagrangians can be displaced by a Hamiltonian isotopy. A related question in contact geometry is the non-squeezing problem [11] or, more quantitatively, the question of constructing capacities [54]. A comparison of the latter with the Floer-theoretic approach can be found in [32].

(b) One strategy for proving mirror symmetry is to apply [17] and reduce the comparison of quasi-coherent sheaves with Lagrangians to a comparison with constructible

[1] We make no claim of exhaustiveness; the author simply discusses here what he is most familiar with.

[2] arXiv preprint, 2008.

sheaves. In the toric case, this is realized as the coherent-constructible correspondence, initiated by Fang, Liu, Treumann, and Zaslow [12]. A complete proof was first given by Kuwagaki [33], with independent proofs around the same time by Vaintrob [50] and, in the proper case, by Zhou [55]. (See [5] for a modern account.) The geometric ingredients needed to apply [17] are obtained in [14].

(c) There is a technique called persistent homology in the applied mathematics field of topological data analysis, invented to detect noise and small changes that occur abruptly. Kashiwara and Schapira [27] discovered a sheaf-theoretic interpretation of such structure. This notion was subsequently used to define a metric, the interleaving distance, on the category of sheaves by Asano and Ike [2] and by Guillermou and Viterbo [20]. This metric, when restricted to a reasonable subcategory, is complete and can be used to study C^0 symplectic and contact geometry [3, 4].

(d) Noncommutative geometry pursues the idea of treating a linear category not necessarily with a symmetric product as a kind of "noncommutative" space, and of developing in this setting the usual structures such as Serre duality, spherical functors, and Calabi–Yau structures; it is a particularly useful framework when studying Fukaya categories. By passing to sheaf-theoretic models, one can attempt to realize these structures there. This strategy has been employed by Shende and Takeda [47] to obtain a smooth Calabi–Yau structure. In a series of papers with Li [31, 29, 30], the author gives an independent construction.

2. Six-functor in topology

The goal of the first half of Part 1 is to construct the six-functor formalism in topology. Roughly speaking, we want an assignment $X \mapsto \mathrm{Sh}(X)$, from a topological space to its category of sheaves, to have two kinds of functoriality. Let $f\colon X \to Y$ be a continuous map. Since sheaves are supposed to categorify functions, or more precisely, cohomology, there should be a notion of pullback $f^*\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$. In the cohomological setting, with X and Y nice enough and f proper, one can also push forward along f by "integration along the fiber." With sheaves, the situation is even better: even without assuming f to be proper, there always exists a proper pushforward $f_!\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. Finally, just as one can multiply two functions when they take values in a ring, when the coefficient category carries a symmetric monoidal product, there is a tensor structure $\otimes$ on $\mathrm{Sh}(X)$. These constitute the basic three functors; the remaining three can be obtained by passing to right adjoints.

Approximate Definition 2.1. Fix some symmetric monoidal category $\mathcal{V}$, which one can for simplicity take it as the category of chain complexes $\mathrm{Mod}_{\mathbb{Z}}$. The six-functor formalism is a functor sending a nice topological space X to the category $\mathrm{Sh}(X)$ of $\mathcal{V}$-valued sheaves on X, equipped with a symmetric monoidal structure $\otimes$ and two kinds of functoriality: for a map $f\colon X \to Y$, there is a $*$-pullback $f^*\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ and a !-pushforward $f_!\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, where $*$ is usually read as "star" and ! is read as "shriek." Furthermore, there are adjunctions

$$f_! \dashv f^!, \quad f^* \dashv f_*, \quad \text{and} \quad F \otimes (-) \dashv \mathcal{H}\mathrm{om}(F, -),$$

where $F \in \mathrm{Sh}(X)$ is a sheaf. In addition, they should satisfy the following compatibilities:

(1) $p_! = p_*$ when p is proper, and $j^! = j^*$ when j is an open embedding.
(2) $*$-pullback is symmetric monoidal.
(3) !-pushforward satisfies the projection formula.

(4) $*$-pullback and !-pushforward satisfy base change.

We explain what the compatibilities listed above mean. First, if $f\colon X \to Y$ is a map and $F, G \in \mathrm{Sh}(Y)$ are sheaves, then f^* being symmetric monoidal means that there is an isomorphism, functorial in F and G, such that

$$f^*(F \otimes G) \cong (f^*F) \otimes (f^*G),$$

and f^* sends the unit to the unit. In particular, since $\mathrm{Sh}(*) = \mathcal{V}$, if we set $1_X := a^*(1)$, where $a\colon X \to \{*\}$ is the projection to a point, then 1_X is the unit of $\mathrm{Sh}(X)$.

For the projection formula, let $F \in \mathrm{Sh}(X)$ and $G \in \mathrm{Sh}(Y)$; then there is an isomorphism

$$f_!(F \otimes f^*G) \cong f_!(F) \otimes G.$$

Finally, to state base change, consider a Cartesian square:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Then, for $F \in \mathrm{Sh}(X)$, there is an isomorphism

$$g^* f_! F \cong f'_! g'^* F.$$

Example 2.2 (The Künneth formula)**.** Assume X is a locally compact Hausdorff space and consider sheaves valued in abelian groups Ab. A classical fact is that, in the derived setting, $\Gamma(X;\mathbb{Z}) \equiv a_*(\mathbb{Z}_X) = C^*(X;\mathbb{Z})$ is the cochain complex of singular cochains. Similarly, $\Gamma_c(X;\mathbb{Z}) \equiv a_!(\mathbb{Z}_X) = C^*_c(X;\mathbb{Z})$ is the complex of compactly supported singular cochains.

Now let X_1 and X_2 be compact and consider the Cartesian square:

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{p_2} & X_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle a_2} \\ X_1 & \xrightarrow{a_1} & \{*\} \end{array}$$

Since X_1 and X_2 are compact, all maps in the diagram are proper, so compatibility (1) of Approximate Definition 2.1 gives $f_! = f_*$ for each map f in the diagram. We find that the six-functor formalism recovers the Künneth formula:

$$\begin{aligned} C^*(X_1;\mathbb{Z}) \otimes C^*(X_2;\mathbb{Z}) &= a_{1*}(\mathbb{Z}_{X_1}) \otimes a_{2*}(\mathbb{Z}_{X_2}) \\ &= a_{1*}(\mathbb{Z}_{X_1} \otimes a_1^* a_{2*}(\mathbb{Z}_{X_2})) \\ &= a_{1*}(\mathbb{Z}_{X_1} \otimes p_{1*} p_2^*(\mathbb{Z}_{X_2})) \\ &= a_{1*} p_{1*}(p_1^*(\mathbb{Z}_{X_1}) \otimes p_2^*(\mathbb{Z}_{X_2})) \\ &= a_{1*} p_{1*}(\mathbb{Z}_{X_1 \times X_2}) \\ &= \Gamma(X_1 \times X_2;\, \mathbb{Z}_{X_1 \times X_2}) \\ &= C^*(X_1 \times X_2;\mathbb{Z}). \end{aligned}$$

Example 2.3 (The Poincaré duality)**.** Let $f : X \to Y$ be a map and set $\omega_f := f^!(\mathbb{Z}_Y)$. We claim that, for any $F \in \mathrm{Sh}(Y)$, there is a canonical map $f^*(F) \otimes \omega_X \to f^!(F)$. Since $f^!$ is a right adjoint, such a map corresponds to a map $f_!(f^*(F) \otimes \omega_X) \to F$. This latter map is given by the composition

$$f_!(f^*(F) \otimes \omega_X) = F \otimes (f_!\omega_X) = F \otimes (f_! f^! \mathbb{Z}_Y) \to F \otimes \mathbb{Z}_Y = F.$$

Here, the first equality is the projection formula and the only map that is not an isomorphism is given by the counit $f_! f^! \mathbb{Z}_Y \to \mathbb{Z}_Y$.

It is a classical fact that, when X and Y are manifolds and $f : X \to Y$ is smooth, i.e., a submersion, then the map is an isomorphism and ω_f is an invertible local system. In particular, when $Y = \{*\}$, $\omega_X \coloneqq a^!(\mathbb{Z})$ is called the dualizing sheaf and is concretely given by $\omega_X = \mathfrak{or}_X[\dim X]$, the orientation sheaf shifted by the dimension of X. Thus, when X is orientable, $f^! = f^*[\dim X]$. But then, if we further assume that X is compact, we obtain the Poincaré duality:

$$\begin{aligned} C^*(X;\mathbb{Z})^\vee \coloneqq \operatorname{Hom}(a_*(\mathbb{Z}_X),\mathbb{Z}) &= \operatorname{Hom}(a_!(\mathbb{Z}_X),\mathbb{Z}) = \operatorname{Hom}(\mathbb{Z}_X, a^!\mathbb{Z}) \\ &= \operatorname{Hom}(a^*\mathbb{Z}, \mathbb{Z}[\dim X]) = a_*(\mathbb{Z}_X)[\dim X] = C^*(X;\mathbb{Z})[\dim X]. \end{aligned}$$

Remark 2.4 (The Verdier duality). The object ω_X is first considered by Verdier. Because of the above Example 2.3, the morphism

$$\begin{aligned} \mathrm{D}_X : \operatorname{Sh}(X) &\to \operatorname{Sh}(X)^{\mathrm{op}} \\ F &\mapsto \mathcal{H}\mathrm{om}(F, \omega_X) \end{aligned}$$

is often referred to as the Verdier duality and D_X the Verdier dual of F. Despite the name, it is only an isomorphism when restricting to objects with some finiteness condition. However, we will see a modern interpretation[3] later that does come from an equivalence.

2.1. **Some higher category background.** We will summarize the needed materials from higher category theory in order to develop sheaf theory. As our goal is to use the theory to talk about things instead of developing it, we will follow the practice of being agnostic of the exact model we use for our ∞-categories, and refer, for example, to Lurie's work [34] for the exact definition. Before we start, we first mention some motivation of why the classical approach of derived category is not the most convenient.

Example 2.5 ([49])**.** The derived category of an abelian category might not have limits and colimits. Consider for example the bounded derived category $D^b(\mathbb{Z})$ of abelian groups. We first recall that there is a non-trivial map $e : \mathbb{Z}/2 \to \mathbb{Z}/2[1]$. Indeed,

$$\operatorname{Hom}_{D^b(\mathbb{Z})}(\mathbb{Z}/2, \mathbb{Z}/2[1]) \cong H^1 \operatorname{RHom}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \operatorname{Ext}^1(\mathbb{Z}/2,\mathbb{Z}/2),$$

and the latter can be computed by resolving $\mathbb{Z}/2$ with the free resolution $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2$. Hom-ing it into $\mathbb{Z}/2$, we see that $\operatorname{RHom}(\mathbb{Z}/2,\mathbb{Z}/2)$ can be computed by the chain complex $\mathbb{Z}/2 \xrightarrow{2=0} \mathbb{Z}/2$ so $\operatorname{Ext}^1(\mathbb{Z}/2,\mathbb{Z}/2) = \mathbb{Z}/2$ has exactly one non-zero element. Now the claim is that $K \coloneqq \ker(e)$ does not exist. Assume otherwise, then we have the following pullback diagram in $D^b(\mathbb{Z})$:

$$\begin{array}{ccc} K & \xrightarrow{\quad i \quad} & \mathbb{Z}/2 \\ \downarrow & & \downarrow{\scriptstyle e} \\ 0 & \xrightarrow{\qquad} & \mathbb{Z}/2[1]. \end{array}$$

[3] Theorem **??**.

Since $H^n(-) \cong \mathrm{Hom}_{D^b(\mathbb{Z})}(\mathbb{Z}[-n], -)$ is limit-preserving for all $n \in \mathbb{Z}$, the same pullback diagram produces a family of short exact sequences

$$0 \to H^n(K) \xrightarrow{H^n(i)} H^n(\mathbb{Z}/2) \xrightarrow{H^n(e)} H^n(\mathbb{Z}/2[1]) \cong H^{n+1}(\mathbb{Z}/2).$$

This implies that i is a quasi-isomorphism so e must be zero, which is a contradiction, implying that K does not exist.

Example 2.6. Let X be a topological space. The abelian category $\mathrm{Loc}(X)$ satisfies descent but its derived category $D^b_{loc}(X)$ doesn't. For example, consider $X = S^2$ and cover it with two open sets U_+ and U_- such that U_+ and U_- are contractible with $U_+ \cap U_- \sim S^1$ homotopic to a circle. As local systems are equivalent to representation of the fundamental group, i.e., $\mathrm{Loc}(X) \cong \mathrm{Fun}(\pi_0(X, x_0), \mathrm{Ab})$, we conclude all three categories $\mathrm{Loc}(S^2) \cong \mathrm{Loc}(U_+) \cong \mathrm{Loc}(U_-) \cong \mathrm{Ab}$ are equivalent to abelian groups. Furthermore, the restriction $\mathrm{Loc}(U_+) \to \mathrm{Loc}(U_+ \cap U_-)$ is fully faithful. For example,

$$\mathrm{Hom}_{\mathrm{Loc}(S^1)}(\mathbb{Z}_{S^1}, \mathbb{Z}_{S^1}) \cong C^0(S^1; \mathbb{Z}) = \mathbb{Z}.$$

As a result, the diagram of restrictions

$$\begin{array}{ccc} \mathrm{Loc}(S^2) & \xrightarrow{\quad i \quad} & \mathrm{Loc}(U_+) \\ \downarrow & & \downarrow e \\ \mathrm{Loc}(U_-) & \xrightarrow{\quad\quad} & \mathrm{Loc}(U_+ \cap U_-). \end{array}$$

is a pullback. On the other hand, the same diagram with $\mathrm{Loc}(-)$ replaced by $D^b_{loc}(-)$ is not a pullback. For example, being a pullback in particular implies that morphisms glue locally. However, since $\mathrm{RHom}_{D^b_{loc}(S^2)}(\mathbb{Z}_{S^2}, \mathbb{Z}_{S^2}) \cong C^*(S^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}[-2]$, there is a non-trivial map $e : \mathbb{Z}_{S^2} \to \mathbb{Z}_{S^2}[2]$. On the other hand, since U_+ and U_- are both contractible, $D^b_{loc}(U_+) \cong D^b_{loc}(U_-) \cong D^b(\mathbb{Z})$, so the restriction $e|_{U_+}$ and $e|_{U_-}$ are both 0. This implies that the same diagram with $D^b_{loc}(-)$ is not a pullback, as otherwise we would conclude that $e = 0$.

The problem with the above two examples is that passing to the homotopy categories forgets too much information: in the derived category, two maps $f, g : X^\bullet \to Y^\bullet$ are the same if and only if they are homotopic, but the homotopy that witnesses their equivalence is a datum that has been thrown away. To maintain the knowledge needed to re-glue the map in Example 2.6, one has to include those homotopies back. However, keeping only them will not produce a good theory when needing to identify homotopies, so one has to maintain all the higher homotopies as well.

One solution is then to replace the building blocks of the world with a homotopical version of sets, and one choice are the (∞-)groupoids, categories whose morphisms are all invertible. These objects can be modeled by topological spaces and are thus called spaces in [34]. There are also attempts to treat them as an alternative foundation rather than set theory and, in these frameworks, they are referred to as homotopy types, where the term "type" is used in the sense of mathematical logic [13] instead of topology. But as the lecture is given at the University of Bonn, we will call them *animated sets* or just *anima*.

Notation 2.7. As we will take higher category theory as our foundation, we will from now on drop the prefix ∞ and simply refer to ∞-categories as categories and refer to the usual categories as classical categories. Similarly, we will refer to ∞-groupoids as groupoids and refer to the usual groupoids as 1-groupoids.

Now, a category $\mathcal{C}$ consists of a collection of objects $\mathrm{Obj}(\mathcal{C})$ and, for a pair of objects X and Y, there is an anima $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of morphisms from X to Y. This means that arrows of the form $f : X \to Y$ give "points" in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. Given two morphisms, $f, g : X \to Y$, there is an anima $\mathrm{Hom}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}(f,g)$ of invertible 2-morphisms from f to g. Naively speaking, a "point" $T : f \Rightarrow g$ is an identification, so an invertible arrow, witnessing the "sameness" of f and g, and the process goes on for n-morphisms for any $n \geq 1$. As before, there is a composition map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) &\to \mathrm{Hom}_{\mathcal{C}}(X,Z)\\ (f,g) &\mapsto g \circ f.\end{aligned}$$

But because of the existence of higher morphisms, associativity of composition for example is now structure instead of a property. That is, for three morphisms $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$, it is part of the data that there is $H \in \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{C}}(X,W)}(h \circ (g \circ f), (h \circ g) \circ f)$ such that

$$H : h \circ (g \circ f) = (h \circ g) \circ f.$$

To relate these different identifications, they are higher identifications, etc. This is often referred to as *coherence structure.*

Example 2.8. Consider a topological space X. One can view its points in X as objects in a category. For two points p and q, morphisms $p \to q$ are given by paths γ from p to q. A 2-morphism between two paths is a homotopy of paths, and so on. Note that, in this case all 1-morphisms are already invertible, and the associated category, by abusing the notation, also denoted by X, is a groupoid.

Even more special, one can consider the case when $X = S^1$ is given by the circle. In this case, since S^1 is path-connected, any two points are the same, but there is a $\mathbb{Z}$-worth of ways to identify the two points, as the fundamental group $\pi_1(S^1, \{p\}) = \mathbb{Z}$ at any given point $p \in S^1$ is given by the integers. However, the classical result $\pi_n(S^1, \{p\}) = 0$ for $n > 1$ shows that any two such choices are either different or essentially the same. In this language, this is saying that S^1 is in fact the 1-groupoid $B\mathbb{Z}$.

Similar to the situation within categories, the notion of functors also comes with coherent structure now. For example, while a functor still comes with an assignment on objects $\mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ with a map, for any $X, Y \in \mathrm{Obj}(\mathcal{C})$,

$$F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)),$$

the compatibility with composition, the compatibility $F(g) \circ F(f) = F(g \circ f)$ is now a datum for morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$. Similarly so is the associativity, and the identifications between the associativity etc. are now all data that need to be supplied. While this looks cumbersome, keeping track of coherence actually has great advantages. One of them is that there is an abundant supply of well-behaved limits and colimits.

Definition 2.9. Let I be a category. We use the notation $I^{\triangleright}$ to denote the category $I \cup \{\infty\}$ with a final cone point ∞ added to I. A diagram $X_\bullet : I \to \mathcal{C}$ is said to have a limit in $\mathcal{C}$ if there exists an extension $\overline{X}_\bullet : I^{\triangleright} \to \mathcal{C}$ such that $\overline{X}_\bullet|_I = X_\bullet$. In this case, we write $\overline{X}_\infty := \lim_I X_\bullet$ and call it the limit of the diagram $X_\bullet$. Sometimes, we will also use the notation $\lim\limits_{\alpha \in I} X_\alpha$ or simply $\lim_{\alpha} X_\alpha$ when it is natural to index by I. A dual notion of colimit $\operatorname{colim}_{\alpha} X_\alpha$ can be defined similarly.

Lemma 2.10. *The category* Ani *admits all small limits and colimits.*

Proof. [37, 02SX]. □

Remark 2.11. For a category $\mathcal{C}$, the object $\operatorname{colim}_{\alpha} X_\alpha$ along with the maps $X_\beta \to \operatorname{colim}_{\alpha} X_\alpha$, for $\beta \in I$, satisfies the universal property

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim}_{\alpha} X_\alpha, Y) = \lim_{\alpha} \operatorname{Hom}(X_\alpha, Y),$$

for $Y \in \mathcal{C}$, where the limit is now taken in Ani. Thus, it is (homotopically) unique. Similarly, we have

$$\operatorname{Hom}_{\mathcal{C}}(Y, \lim_{\alpha} X_\alpha) = \lim_{\alpha} \operatorname{Hom}(Y, X_\alpha)$$

for any $Y \in \mathcal{C}$.

Notation 2.12. We denote by Cat the (large) category of small categories with morphisms given by functors.

Remark 2.13. The theory we refer to as ∞-category sits in a larger hierarchy of the theory of (n, m)-category. The number n here denotes the degree at which non-trivial morphisms exist and the number m denotes the degree at which all morphisms beyond it should be invertible.

For example, a classical category has non-trivial morphisms but two morphisms f and g are either different or exactly the same in essentially one way, so it can be viewed as a $(1, 1)$-category. Another special case is groupoids, where non-trivial morphisms exist for all degrees but they are always invertible, so a groupoid is the same as an $(\infty, 0)$-category. Their common intersection is a category where two objects X and Y are either strictly the same or different. Thus, up to size issue, a $(1, 0)$-category is just a set.

What we call an ∞-category is then an $(\infty, 1)$-category. The category Cat is most naturally an $(\infty, 2)$-category, since it is natural to include natural transformations that are not invertible, for example, for the purpose of adjoint functors. We will, however, keep this point implicit.

Proposition 2.14. *The category* Cat *admits all small limits and colimits.*

Proof. [37, Tag 02T0 and Tag 02UN]. □

Remark 2.15. Limits in Cat are relatively easy to compute. Let $\mathcal{C}_\alpha : I \to \mathrm{Cat}$ be a diagram of categories. Then objects in $\mathcal{C} := \lim_{\alpha} \mathcal{C}_\alpha$ are given by compatible families of the form $(x_\alpha)_{\alpha \in I}$ where, for each $\alpha \in I$, x_α is an object of $\mathcal{C}_\alpha$. Given two such families $(x_\alpha)_{\alpha \in I}$, $(y_\alpha)_{\alpha \in I}$, the hom-anima is simply given by the limit hom-anima

$$\operatorname{Hom}_{\mathcal{C}}\left((x_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I}\right) = \lim_{\alpha} \operatorname{Hom}_{\mathcal{C}_\alpha}(x_\alpha, y_\alpha).$$

Unlike limits, colimits are often harder to compute. However, in the case of filtered colimits, there is a simple description.

Proposition 2.16. *Let* $\mathcal{C}_\alpha : I \to \mathrm{Cat}$ *be a filtered diagram of categories with transition map* $f_{\beta\alpha} : \mathcal{C}_\alpha \to \mathcal{C}_\beta$ *and denote by* $\mathcal{C} := \operatorname{colim}_{\alpha \in I} \mathcal{C}_\alpha$ *their colimit, which comes with a canonical map*

$f_\alpha : \mathcal{C}_\alpha \to \mathcal{C}$. Then, objects in $\mathcal{C}$ are given by objects $f_\alpha(x_\alpha)$ for $x_\alpha \in \mathcal{C}_\alpha$. Furthermore, given $x_\alpha \in \mathcal{C}_\alpha$ and $y_\beta \in \mathcal{C}_\beta$, the hom-anima can be computed as

$$\mathrm{Hom}_{\mathcal{C}}(f_\alpha(x_\alpha), f_\beta(y_\beta)) = \operatorname*{colim}_{\alpha,\beta\to\gamma} \mathrm{Hom}_{\mathcal{C}_\gamma}(f_{\gamma\alpha}(x_\alpha), f_{\gamma\beta}(y_\beta))$$

where γ runs over all indices more final than α and β.

Proof. [44]. □

As a special case of Proposition 2.14, the category Cat admits Cartesian product: For $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$, there exists a product category $\mathcal{C} \times \mathcal{D}$. In fact, it can be organized into a closed symmetric monoidal structure on Cat, usually referred to as the Cartesian monoidal structure. The internal Hom in this case is given by the functor category: For $\mathcal{C}$, $\mathcal{D} \in \mathrm{Cat}$, the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$ with morphisms given by natural transformations satisfies the property that, for any category $\mathcal{E} \in \mathrm{Cat}$, we have

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) = \mathrm{Hom}_{\mathrm{Cat}}\left(\mathcal{E}, \mathrm{Fun}(\mathcal{C}, \mathcal{D})\right).$$

Similar to the classical situation, the Yoneda lemma is true: Let $\mathcal{C}$ be a category. Consider the assignment $X \mapsto h_X$ where $h_X \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ani})$ is the corepresentable functor corepresenting X, $h_X(Y) \coloneqq \mathrm{Hom}(Y, X)$.

Remark 2.17. There is a size issue here since the category Ani is not small; the collection of its objects, containing the collection of all sets, forms a proper class rather than just a set. The usual way around this issue is to fix two universes $\mathcal{U} \subseteq \mathcal{V}$ at the beginning, and declare that a set is small if it lies in $\mathcal{U}$, is large if it lies in $\mathcal{V}$, and it is very large if it lies outside $\mathcal{V}$. This way, Cat consists of those $\mathcal{C}$ such that $\mathrm{Obj}(\mathcal{C}) \in \mathcal{U}$, and for Ani, we mean the category of anima in $\mathcal{U}$, so the category lies in $\mathcal{V}$, and is hence a large category. For this reason, we will consider the very large category $\widehat{\mathrm{Cat}}$ whose objects are large categories.

Definition 2.18. To simplify the notation, we will write $\mathrm{PSh}(\mathcal{C}) \coloneqq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ani})$ and call it the category of presheaves on $\mathcal{C}$. As discussed in Remark 2.17, $\mathrm{PSh}(\mathcal{C}) \in \widehat{\mathrm{Cat}}$ is a large category.

Theorem 2.19. *The assignment $(X \mapsto h_X)$ organizes into an embedding $h_\bullet : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$.*

(1) (The strong Yoneda lemma) Assume $\mathcal{C}$ is locally small. That is, $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is small for all $X, Y \in \mathcal{C}$. Then, for any $F \in \mathrm{PSh}(\mathcal{C})$, evaluation induces the equivalence

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_X, F) = F(X).$$

(2) If $\mathcal{C}$ is small, then $h_\bullet$ exhibits $\mathrm{PSh}(\mathcal{C})$ as the free cocompletion of $\mathcal{C}$.

Here (2) means that $\mathrm{PSh}(\mathcal{C})$ admits all small colimits and, for any category $\mathcal{E}$ with all small colimits, the restriction map

$$\mathrm{Fun}^{\mathrm{cocont}}(\mathrm{PSh}(\mathcal{C}), \mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}_0, \mathcal{E})$$

is an equivalence, where we use $\mathrm{Fun}^{\mathrm{cocont}}(-, \mathcal{E}) \subseteq \mathrm{Fun}(-, \mathcal{E})$ to denote the subcategory of functors that preserve all small colimits.

Proof. One can trace the details backwards starting from [37, Tag 03V7]; the strong Yoneda lemma is [37, Tag 03M5]. We only remark that $\mathrm{PSh}(\mathcal{C})$ indeed admits all small colimits. In fact, for any category $\mathcal{D}$ with all small colimits, the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ will have all small colimits as they are given pointwise by the formula

$$(\operatorname*{colim}_{\alpha} F_\alpha)(X) \coloneqq \operatorname*{colim}_{\alpha} (F_\alpha(X))$$

for any diagram of functors $F_\cdot : I \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$, where the colimit is taken in $\mathcal{D}$. The same argument shows that the same statement holds for limits. Note that we use the following lemma about anima. □

Remark 2.20. The weak Yoneda lemma is the special case when $F = h_Y$, which implies that

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_X, h_Y) = h_Y(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

so the Yoneda embedding is fully faithful.

Remark 2.21. In fact $\mathrm{PSh}(\mathcal{C})$ also admits all small limits, which can be computed pointwise as well.

2.2. **Presentable categories.** Constructing functors in the higher categorical setting is hard, as it requires specifying infinitely many layers of data. Often, it is easy to begin with a functor that can be constructed classically and use it to build more functors from a standard set of tools. The standard way to begin is through the adjoint functor theorem.

Definition 2.22. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A functor $G : \mathcal{D} \to \mathcal{C}$ is said to be a right adjoint to F if there is a pair of natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ such that, for any $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, the compositions, depending functorially both on X and Y,

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(G(F(X)), G(Y)) \xrightarrow{(-)\circ\eta_X} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

and

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(X), F(G(Y))) \xrightarrow{\epsilon_Y\circ(-)} \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$$

are inverse to each other. In this case, we write $F \dashv G$ and call η the unit and ϵ the counit. As the definition is symmetric, we also say that F is the left adjoint of G.

Remark 2.23. We note that, as being an adjoint is a universal property, once one exists, the functoriality implies that it is unique. Thus, having a left or a right adjoint is a property rather than an extra structure, and, in the above situation, we can write $G = F^R$ or $F = G^L$.

Example 2.24. Consider the forgetful map $G : \mathrm{Ab} \to \mathrm{Set}$, its left adjoint $F : \mathrm{Set} \to \mathrm{Ab}$ is given by taking the free abelian group $F(X) \coloneqq \mathbb{Z}^{\oplus X}$ for $X \in \mathrm{Set}$. Denote the basis of $\mathbb{Z}^{\oplus X}$ by e_x, for $x \in X$, the unit is given by

$$\begin{aligned} \eta_X : X &\to \mathbb{Z}^{\oplus X} \\ x &\mapsto e_x, \end{aligned}$$

and the counit is given by

$$\begin{aligned} \epsilon_M : \mathbb{Z}^{\oplus M} &\to M \\ e_m &\mapsto m. \end{aligned}$$

Remark 2.25. More concretely, the requirement for $F^L \dashv F$ to be an adjunction is that the pair ϵ and η satisfies the triangle identity: For any $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, the composition

$$F(X) \xrightarrow{F(\eta_X)} FF^LF(X) \xrightarrow{\epsilon_{F(X)}} F(X)$$

and

$$F^L(Y) \xrightarrow{\eta_{F^L(Y)}} F^L F F^L(Y) \xrightarrow{F^L(\epsilon_Y)} F^L(Y)$$

are equivalent to identities. In particular, whether a pair of functors F and G is an adjunction can be checked in the homotopy category [43, Remark 4.4.5].

Example 2.26 ([37, Tag 02ZZ])**.** Let $f : \mathcal{C}_0 \to \mathcal{D}_0$ be a functor between two small categories and $\mathcal{E}$ a category with colimits. Precomposing with f defines a functor

$$(-) \circ f : \mathrm{Fun}(\mathcal{D}_0, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}_0, \mathcal{E})$$

between the functor categories. This functor has a left adjoint $\mathrm{Lan}_f(-)$ which on a functor $G : \mathcal{C}_0 \to \mathcal{E}$ is given by, when evaluating at $Y \in \mathcal{D}_0$,

$$\mathrm{Lan}_f(G)(Y) \coloneqq \operatorname*{colim}_{f(X)\to Y} G(X)$$

where the index runs over pairs of $X \in \mathcal{C}_0$ and a morphism $f(X) \to Y$. The $\mathrm{Lan}_f(G) : \mathcal{D}_0 \to \mathcal{E}$ is called the left Kan extension of G along f. Being a left adjoint, it is initial among $H : \mathcal{D}_0 \to \mathcal{E}$ that admits a map $G \to H \circ f$. When $\mathcal{E}$ admits limits, there is a dual notion of right Kan extension $\mathrm{Ran}_f(-)$. Naively on objects, it is given by

$$\mathrm{Ran}_f(G)(Y) \coloneqq \lim_{Y\to f(X)} G(X)$$

Often, for simplicity, one would denote the functor $(-) \circ f$ by f^*, the left Kan extension $\mathrm{Lan}_f(-)$ by $f_!$, and $\mathrm{Ran}_f(-)$ by f_*.

Theorem 2.27 (The Adjoint Functor Theorem)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable categories. Then,*

(1) A functor $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if it preserves colimits.
(2) A functor $G : \mathcal{D} \to \mathcal{C}$ is a right adjoint if and only if it is accessible and preserves limits.

Proof. [37, Tag 06Q4]. □

Here, presentability is an assumption on the size of the category. Roughly speaking, we want to mimic Example 2.26 and define the right adjoint of F by the formula,

$$F^R(Y) \coloneqq \lim_{Y\to F(X)} X$$

so it would be nice for $\mathcal{C}$ to admit limits. But further examining the formula, one realizes that the limit diagram, which takes over all $X \in \mathcal{C}$ that admits a map from Y upon mapping to $\mathcal{D}$ by F, is very likely not small if $\mathcal{C}$ is large enough to have all small limits. The solution is to consider categories large enough to have limits and colimits but is essentially controlled by a small amount of data, so one can reduce the size of the diagram. Such categories are in fact very natural.

Example 2.28. We consider an example from classical category theory. Denote by Ab the category of abelian groups. We recall that any abelian group M is a filtered colimit of finitely generated abelian groups, through, for example, the inclusion of finitely generated subgroups.

Furthermore, such finitely generated abelian groups are "small" in the following categorical sense: If M is finitely generated, then for any filtered diagram $\{M_\alpha\}$, the canonical map

$$\operatorname*{colim}_{\alpha} \mathrm{Hom}_{\mathrm{Ab}}(M,\ M_\alpha) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ab}}(M, \operatorname*{colim}_{\alpha} M_\alpha)$$

is an isomorphism. Indeed, the statement is true for M of the form $\mathbb{Z}^k$, $k \in \mathbb{N}$, but any finitely generated abelian group fits into a short exact sequence $0 \to \mathbb{Z}^l \to \mathbb{Z}^k \to M \to 0$.

Definition 2.29. In this definition, we use $\mathcal{C}$ to denote a category and κ a small regular cardinal.

(1) ([37, Tag 02P8]) A category I is κ-filtered if any κ-small diagram $K \to I$ admits an extension to $K^{\triangleright} \to I$.
(2) ([37, Tag 0649]) Assume $\mathcal{C}$ admits κ-filtered colimits. An object $X \in \mathcal{C}$ is κ-compact if the functor $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{Ani}$ preserves κ-filtered colimits. More concretely, if $Y_\alpha : I \to \mathcal{C}$ is a diagram such that I is κ-filtered, then the canonical map

$$\operatorname*{colim}_{\alpha} \mathrm{Hom}_{\mathcal{C}}(X, Y_\alpha) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, \operatorname*{colim}_{\alpha} Y_\alpha)$$

is an isomorphism.
(3) ([37, Tag 0673]) A category $\mathcal{C}$ is κ-compactly generated if every object of $\mathcal{C}$ is a colimit of κ-compact objects.

Remark 2.30. Let κ be a cardinal. A set X is said to be κ-small if $|X| < \kappa$. A cardinal κ is regular if for any κ-small index set I and any collection $\{X_\alpha\}_{\alpha \in I}$ of κ-small sets, the union $\bigcup X_\alpha$ is κ-small.

Example 2.31. The smallest infinite regular cardinal is aleph zero, $\aleph_0 := |\mathbb{N}|$, as a set X is $\aleph_0$-small by definition if and only if it is finite. Because of this, one can check that $\aleph_0$-filtered recovers the usual definition of being filtered. For this reason, we follow the convention of referring to an $\aleph_0$-compactly generated category simply as a compactly generated category.

Definition 2.32 ([37, Tag 06GT])**.** Let $\mathcal{C}$ be a category.

(1) We say $\mathcal{C}$ is κ-accessible if it is κ-compactly generated and the full subcategory $\mathcal{C}^\kappa \subseteq \mathcal{C}$ of κ-compact objects is small.
(2) We say $\mathcal{C}$ is accessible if it is κ-accessible for some small regular cardinal κ.

Remark 2.33. We note that there is an inconsistency in the conventions of compactly generated in Example 2.31 and accessibility in Definition 2.32.

Notation 2.34. As in Example 2.31, when $\mathcal{C}$ is compactly generated, one often denotes the subcategory of compact objects simply by $\mathcal{C}^c$ or $\mathcal{C}^\omega$ instead of $\mathcal{C}^{\aleph_0}$.

We gave an intrinsic definition for accessibility above. However, it is shown in [37, 06K7] that there is also an extrinsic definition: A category $\mathcal{C}$ is κ-accessible if it is of the form $\mathrm{Ind}_\kappa(\mathcal{C}_0)$, whose definition we now recall for later use.

Definition 2.35 ([37, Tag 063J])**.** For a small category $\mathcal{C}_0$ and small regular cardinal κ, the Ind_κ-completion $\mathrm{Ind}_\kappa(\mathcal{C}_0)$ is obtained by freely joining κ-filtered colimits to $\mathcal{C}_0$. More precisely, there is a fully faithful map $i : \mathcal{C}_0 \hookrightarrow \mathrm{Ind}_\kappa(\mathcal{C}_0)$ such that for any $\mathcal{E}$ with κ-filtered colimits, the restriction map

$$\mathrm{Fun}^{\kappa\text{-}\mathrm{fin}}(\mathrm{Ind}_\kappa(\mathcal{C}_0), \mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}_0, \mathcal{E})$$

is an equivalence. Here we use $\mathrm{Fun}^{\kappa\text{-}\mathrm{fin}}(-, \mathcal{E}) \subseteq \mathrm{Fun}(-, \mathcal{E})$ to denote the subcategory of functors that preserve κ-filtered colimits.

Remark 2.36. Let $\mathcal{C}_0$ be a small category. As explained in [37, Tag 04BH], Ind_κ-completion can be constructed as the smallest category in $\mathrm{PSh}(\mathcal{C}_0)$ containing corepresentable functors, the image of $\mathcal{C}_0$ under the Yoneda embedding, and closed under taking κ-filtered colimits. Thus, the filtered colimit $\operatorname*{colim}_\alpha h_{X_\alpha}$ taken in $\mathrm{PSh}(\mathcal{C}_0)$, which is often written as the formal filtered colimit "$\operatorname*{colim}_\alpha$"$X_\alpha$, are objects of $\mathrm{Ind}_\kappa(\mathcal{C}_0)$. As explained in [37, Tag 065H], all objects in $\mathrm{Ind}_\kappa(\mathcal{C}_0)$ are of this form. This fact is very useful since we can then compute morphisms more easily. Let "$\operatorname*{colim}_\alpha$" X_α and "$\operatorname*{colim}_\beta$" Y_β be objects of $\mathrm{Ind}_\kappa(\mathcal{C}_0)$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(\text{“}\operatorname*{colim}_\alpha\text{”}\, X_\alpha, \text{“}\operatorname*{colim}_\beta\text{”}\, Y_\beta) &= \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}_0)}(\text{“}\operatorname*{colim}_\alpha\text{”}\, X_\alpha, \text{“}\operatorname*{colim}_\beta\text{”}\, Y_\beta)\\ &= \lim_\alpha \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}_0)}(X_\alpha, \text{“}\operatorname*{colim}_\beta\text{”}\, Y_\beta)\\ &= \lim_\alpha \operatorname*{colim}_\beta \mathrm{Hom}_{\mathcal{C}_0}(X_\alpha, Y_\beta).\end{aligned}$$

Here, the second equality follows from the universal property of the colimit and the last equality is the Yoneda lemma.

Example 2.37. Example 2.28 implies that the category Ab is the Ind-completion of the category of finitely generated abelian groups. One can show that similarly, the category of sets Set is the Ind-completion of finite sets, and the category of rings is the Ind-completion of finitely generated rings.

Example 2.38 ([37, Tag 06GV])**.** The category Ani is compactly generated and the compact objects are given by finitely dominated anima Ani^c. This is the subcategory generated by the point $* \in \mathrm{Ani}$ under finite colimits and retractions. From the spaces viewpoint, $X \in \mathrm{Ani}^c$ if there is a finite CW-complex K such that X is a retract of K.

The definition of accessibility takes care of the "controlled by a small amount of data" part. We also require the part that $\mathcal{C}$ admits both limits and colimits. In fact, requiring one automatically gives the other.

Definition 2.39 ([37, Tag 06NC])**.** A category $\mathcal{C}$ is presentable if it is accessible and furthermore admits small colimits.

Proposition 2.40 ([37, Tag 06PU])**.** *Assume a category $\mathcal{C}$ is accessible. Then, $\mathcal{C}$ admits small colimits if and only if it admits small limits.*

Corollary 2.41. *Let $\mathcal{C}$ be a small category. Then $\mathrm{PSh}(\mathcal{C})$ is presentable.*

Proof. We first observe that $\mathrm{PSh}(\mathcal{C}_0)$, as explained in Theorem 2.19, admits small colimits which are computed pointwise, and thus the corepresentable functors are compact objects. Now, in the same tag [37, Tag 065H] cited in Remark 2.36, it is explained that all objects can be obtained as colimits of corepresentable functors. In other words, the naive equivalence

$$\operatorname*{colim}_{h_X \to T} h_X \xrightarrow{\sim} T$$

can be made precise after carefully choosing a κ. ☐

Definition 2.42 ([37, Tag 06KX])**.** We say a functor $F : \mathcal{C} \to \mathcal{D}$ between accessible categories is accessible if F preserves κ-filtered colimits for some κ.

For the purpose of later constructions, we mention that the collection of presentable categories also forms a nice category.

Definition 2.43. We denote by $\mathrm{Pr}^{\mathrm{L}} \subseteq \widehat{\mathrm{Cat}}$ the non-full subcategory consisting of presentable categories with morphisms given by left adjoints. Similarly, we denote by $\mathrm{Pr}^{\mathrm{R}} \subseteq \widehat{\mathrm{Cat}}$ the non-full subcategory with morphisms given by right adjoints.

Proposition 2.44. *There is an equivalence* $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} = \mathrm{Pr}^{\mathrm{R}}$ *which is given by the identity on objects and by passing to adjoints on morphisms. Furthermore, both* Pr^{L} *and* Pr^{R} *admit all small limits and the inclusions* $\mathrm{Pr}^{\mathrm{L}}, \mathrm{Pr}^{\mathrm{R}} \subseteq \widehat{\mathrm{Cat}}$ *preserve them.*

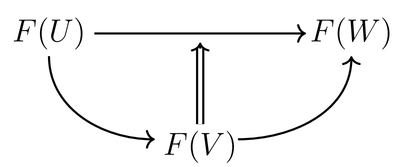
Proof. The equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} = \mathrm{Pr}^{\mathrm{R}}$ is explained in [37, Tag 06Q9]. The existence of limits for Pr^{L} and Pr^{R} and the preservation of them when included in $\widehat{\mathrm{Cat}}$ is explained in [37, Tag 06PH]. □

Remark 2.45. Proposition 2.44 implies that Pr^{L} (and similarly Pr^{R}) also admits colimits: Consider a diagram $\mathcal{C}_\bullet : I \to \mathrm{Pr}^{\mathrm{L}}$ with transition maps given by $F_{\beta\alpha} : \mathcal{C}_\alpha \to \mathcal{C}_\beta$. The colimit $\operatorname*{colim}_I \mathcal{C}_\bullet$ in Pr^{L}, using $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} = \mathrm{Pr}^{\mathrm{R}}$, can be computed as the limit $\lim_{I^{\mathrm{op}}} \mathcal{C}_\bullet$ in Pr^{R}, where we change the diagram to $\mathcal{C}_\bullet : I^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}$ with the same vertices but the morphisms are reversed to $F^R_{\beta\alpha} : \mathcal{C}_\beta \to \mathcal{C}_\alpha$. In this case, the limit can be computed naively in Cat by Remark 2.15.

2.3. **Anima-valued sheaves.** Our goal here is to define the notion of sheaves valued in anima, show that it is a presentable category, and study the $*$-adjunction. We mention that in this "unstable" setting[4], the !-functoriality does not exist. Nevertheless, many important properties, proper base-change for example, already exist in this setting.

Definition 2.46. Let X be a topological space. Denote by Op_X the poset of opens in X given by inclusion. For a category $\mathcal{C}$[5], viewed as coefficient, we use the notation $\mathrm{PSh}(X; \mathcal{C}) := \mathrm{Fun}(\mathrm{Op}_X^{\mathrm{op}}, \mathcal{C})$ and call its objects presheaves. When $\mathcal{C} = \mathrm{Ani}$, we would use the notation $\mathrm{PSh}(X) := \mathrm{PSh}(\mathrm{Op}_X) := \mathrm{Fun}(\mathrm{Op}_X^{\mathrm{op}}, \mathrm{Ani})$ for simplicity.

Geometrically, a presheaf F on X is an assignment sending an open set U to an object $F(U) \in \mathcal{C}$ and inclusion $V \subset U$ to a restriction map $F(U) \to F(V)$. As we are in the higher categorical setting, a double inclusion $W \subseteq V \subseteq U$ is assigned to a two morphism



and so on. To detect the topological information of the space, we would like it to respect the topology.

Definition 2.47. A presheaf $F : \mathrm{Op}_X^{\mathrm{op}} \to \mathcal{C}$ is a sheaf if it satisfies the following *local-to-global* property: For any open $U \subseteq X$ and any open cover $\{U_\alpha\}_{\alpha \in I}$, one can form an augmented Čech complex,

$$F(U) \to \prod_{\alpha \in I} F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta \in I} F(U_\alpha \cap U_\beta) \substack{\to\\\to\\\to} \prod_{\alpha,\beta,\gamma \in I} F(U_\alpha \cap U_\beta \cap U_\gamma) \substack{\to\\\to\\\to\\\to} \cdots$$

[4]The term "unstable" is used to emphasize that the situation is different from the stable setup which we will recall in the next Section 2.4.

[5]The "mental drawback" of working in the higher categorical setting is that categories are both the framework and the objects of interest themselves. Thus, I will use the chancery version $\mathcal{C}$ for the first case and the script version $\mathscr{C}$ for the latter.

where the arrows are given by the restrictions, and we require $F(U)$ to be the limit

$$F(U) \xrightarrow{\sim} \lim \left(\prod_{\alpha\in I} F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta\in I} F(U_\alpha \cap U_\beta) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{\alpha,\beta,\gamma\in I} F(U_\alpha\cap U_\beta\cap U_\gamma) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right). \tag{1}$$

Often, we would simply write it as $F(U) \xrightarrow{\sim} \lim_{\alpha\in I} F(U_\alpha)$. We denote by $\mathrm{Sh}(X;\mathcal{C}) \subseteq \mathrm{PSh}(X;\mathcal{C})$ the subcategory of sheaves. Similar to the case of presheaves, when $\mathcal{C} = \mathrm{Ani}$, we will use the notation $\mathrm{Sh}(X) := \mathrm{Sh}(X;\mathrm{Ani})$ for simplicity.

Example 2.48. Consider the case when $\mathcal{C}$ is the classical category of $\mathbb{R}$-vector spaces $\mathrm{Vect}_{\mathbb{R}}^{\heartsuit}$. In this case, the complex in Definition 2.47 truncates at degree two and linearity further simplifies the sheaf condition to requiring the sequence

$$0 \to F(U) \xrightarrow{r_\alpha} \prod_{\alpha\in I} F(U_\alpha) \xrightarrow{r_{\alpha\beta,\alpha} - r_{\beta\alpha,\beta}} \prod_{\alpha,\beta\in I} F(U_{\alpha\beta})$$

to be exact. Here we use the notation $U_{\alpha\beta} = U_\alpha \cap U_\beta$.

More precisely, exactness at $F(U)$ means that two sections s_1, s_2 are equal if $s_1|_{U_\alpha} = s_2|_{U_\alpha}$ for all α, and exactness at $\prod_\alpha F(U_\alpha)$ means that if a family of sections s_α on U_α agrees on overlaps, i.e., $s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}}$, then they come from a global section s on X with $s_\alpha = s|_{U_\alpha}$.

Standard examples of $\mathrm{Vect}_{\mathbb{R}}^{\heartsuit}$ sheaves are given by functions. For example, $\mathbb{R}$-valued continuous function $C^0_X(U) := C^0(U;\mathbb{R})$ with the obvious restriction is such a sheaf. Similarly, if X, for example, has a smooth/real analytic/holomorphic structure, then the assignment to smooth/real analytic/holomorphic functions $C^\infty_X/C^\omega_X/\mathcal{O}_X$ will be a subsheaf of C^0_X. Another more homotopical example is the locally constant function $C^{lc}_X(U) := C^{lc}(U;\mathbb{R}) \cong C^0(U;\mathbb{R}^\delta)$, where $\mathbb{R}^\delta$ is the topological space $\mathbb{R}$ with the discrete topology, and one recalls that in good cases it computes the homology.

We will use the inclusion $\mathrm{Sh}(X) \hookrightarrow \mathrm{PSh}(X)$ to deduce that $\mathrm{Sh}(X)$ is also presentable. The procedure holds more generally: localizing along a family of morphisms in a presentable category will produce a presentable category.

Definition 2.49 ([37, Tag 06UU])**.** Let $\mathcal{C}$ be a presentable category. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is a Bousfield localization functor if $\mathcal{D}$ is presentable and F has a fully faithful right adjoint. [6]

Proposition 2.50 ([37, Tag 06VH])**.** *Let $\mathcal{C}$ be a presentable category and W be a small collection of morphisms, and $\mathcal{C}_0 \subseteq \mathcal{C}$ be the subcategory of W-local objects. Then, $\mathcal{C}_0$ is a Bousfield localization of $\mathcal{C}$.*

Remark 2.51. In fact, $\mathcal{C}_0$ can be characterized as the universal category with morphisms in W being inverted and is thus sometimes denoted by $\mathcal{C}[W]$. That is, there is a left adjoint $L : \mathcal{C} \to \mathcal{C}[W]$ to the inclusion $\mathcal{C}[W] := \mathcal{C}_0 \subseteq \mathcal{C}$ such that, for any category $\mathcal{D}$ with colimits, precomposing with L induces an inclusion

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{C}[W], \mathcal{D}) \hookrightarrow \mathrm{Fun}^{\mathrm{colim}}(\mathcal{C}, \mathcal{D})$$

whose image consists of $F : \mathcal{C} \to \mathcal{D}$ that sends any $w \in W$ to an isomorphism $F(w)$.

[6] This is called accessible localization in [34].

Definition 2.52 ([37, Tag 04KG])**.** Let $w : X \to Y$ be a morphism in $\mathcal{C}$. An object $C \in \mathcal{C}$ is said to be w-local if the induced map

$$(-) \circ w : \mathrm{Hom}_{\mathcal{C}}(Y, C) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, C)$$

is an isomorphism. For a collection of morphisms W, an object $C \in \mathcal{C}$ is said to be W-local if it is w-local for all $w \in W$.

Theorem 2.53. *Let X be a topological space. The inclusion $\mathrm{Sh}(X) \subseteq \mathrm{PSh}(X)$ is a Bousfield localization. In particular, $\mathrm{Sh}(X)$ is presentable.*

Proof. Let $U \subseteq X$ be an open set. The (strong) Yoneda Lemma, (1) of Theorem 2.19, implies that $F(U) = \mathrm{Hom}_{\mathrm{PSh}(X)}(h_U, F)$. Note that this embedding is covariant: if $V \subseteq U$ is an inclusion of opens, then we have the unique map $h_V \to h_U$. Now, let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of U, we see that there is a diagram

$$\cdots \substack{\to\\\to\\\to\\\to} \coprod_{\alpha,\beta,\gamma \in I} h_{U_{\alpha\beta\gamma}} \substack{\to\\\to\\\to} \coprod_{\alpha,\beta \in I} h_{U_{\alpha\beta}} \rightrightarrows \coprod_{\alpha \in I} h_{U_\alpha} \to h_U$$

and thus the morphism

$$\mathrm{colim}\left(\cdots \substack{\to\\\to\\\to\\\to} \coprod_{\alpha,\beta,\gamma \in I} h_{U_{\alpha\beta\gamma}} \substack{\to\\\to\\\to} \coprod_{\alpha,\beta \in I} h_{U_{\alpha\beta}} \rightrightarrows \coprod_{\alpha \in I} h_{U_\alpha}\right) \to h_U \tag{2}$$

at which one evaluates F to get the sheaf condition (1). In other words, the subcategory of sheaves consists exactly of objects local with respect to all such morphisms. □

Definition 2.54. By Theorem 2.53, the inclusion $\mathrm{Sh}(X) \hookrightarrow \mathrm{PSh}(X)$ has a left adjoint and we will refer to it as the sheafification functor. Furthermore, for a presheaf F, we will denote its sheafification by $F^\dagger$; the symbol is pronounced as "dagger".

Remark 2.55. In fact, every presentable category is a Bousfield localization of a presheaf category [37, Tag 06VP]. What is special about the sheafification functor is that it has a very explicit construction [34, Remark 6.2.2.12]. This construction works more generally for sheaves on a site, and in particular shows that the left adjoint is left exact, so sheaves on a site form a topos [34, Proposition 6.2.2.7]. In our case, for a presheaf F, one can consider the assignment

$$F^{\hat{\dagger}}(U) := \mathrm{colim}_{\mathcal{U}} \lim_{U_\alpha \in \mathcal{U}} F(U_\alpha)$$

where the filtered colimit, indexed by $\mathcal{U}$, runs over the family of all open covers $\{U_\alpha\}$ of U and, for a fixed open cover, the limit is given by (1). In the classical situation, when $\mathcal{C} = \mathrm{Set}$, this gives a description of sheafification. When $\mathcal{C} = (1,1)\mathrm{Cat}$ is the $(2,1)$-category of classical categories, one has to do it twice and the sheafification is given by $(F^{\hat{\dagger}})^{\hat{\dagger}}$. The essential content of [34, Proposition 6.2.2.7] is that, in the higher categorical setting, a transfinite induction taking $(-)^{\hat{\dagger}}$ will give the sheafification.

Our next goal is to construct the $*$-functoriality: For a continuous map $f : X \to Y$, we will construct an adjunction $f^* : \mathrm{Sh}(Y) \rightleftharpoons \mathrm{Sh}(X) : f_*$, the $*$-pullback and $*$-pushforward.

Construction 2.56. Constructing pushforward is straightforward. The map f induces a map

$$f^{-1} : \mathrm{Op}_Y \to \mathrm{Op}_X$$
$$V \mapsto f^{-1}(V)$$

between the posets. Applying $\mathrm{PSh}(-)$, we obtain a pushforward

$$f_*^{\mathrm{pre}} : \mathrm{PSh}(X) \to \mathrm{PSh}(Y)$$

on the level of presheaves and on objects it is given by the formula $(f_*F)(V) = F\left(f^{-1}(V)\right)$. The observation is that its restriction to $\mathrm{Sh}(X)$ factors through $\mathrm{Sh}(Y)$, as an open cover of V pulls back to an open cover of $f^{-1}(V)$ and the sheaf condition (1) of F implies that of f_*F. In summary, we have a $*$-pushforward functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

Remark 2.57. Observe that f_*^{pre} is a special example of the functor obtained from precomposition discussed in Example 2.26. However, the notation is somehow reversed since we reverse the arrow one extra time when passing from $f : X \to Y$ to $f^{-1} : \mathrm{Op}_Y \to \mathrm{Op}_X$. Both notations are, unfortunately, standard.

Construction 2.58. As noted in Remark 2.57, as f_*^{pre} is a precomposition, it admits a left adjoint $f^{*,\mathrm{pre}} : \mathrm{PSh}(Y) \to \mathrm{PSh}(X)$ by the left Kan extension. Recall from Example 2.26, on objects, it uses opens $V \subseteq Y$ to approximate opens $U \subseteq X$. (Note however that, since $\mathrm{PSh}(X) = \mathrm{Fun}(\mathrm{Op}_X^{\mathrm{op}}, \mathrm{Ani})$, there is an extra op that flips the direction of the arrows.) More concretely, for $G \in \mathrm{PSh}(Y)$ and $U \subseteq X$, we have

$$f^{*,\mathrm{pre}}(G)(U) = \operatorname*{colim}_{f^{-1}(V) \supseteq U} G(V)$$

where V runs over open sets in Y that contains $f^{-1}(U)$. Equivalently, since $f(U)$ might not be an open set in Y, one runs over all its open neighborhoods V and assigns the anima by those $G(V)$.

Now, as f_* is restricted from f_*^{pre}, the left adjoint $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is given by the composition

$$\mathrm{Sh}(Y) \hookrightarrow \mathrm{PSh}(Y) \xrightarrow{f^{*,\mathrm{pre}}} \mathrm{PSh}(X) \xrightarrow{(-)^\dagger} \mathrm{Sh}(X).$$

Example 2.59. Let $i : S \subseteq X$ be a subset and we endow it with the subspace topology. For a presheaf $G \in \mathrm{PSh}(X)$, we claim that the counit $i^{*,\mathrm{pre}} i_*^{\mathrm{pre}} \to \mathrm{id}$ is an isomorphism. In particular, if $G \in \mathrm{Sh}(S)$ is a sheaf, then $i^{*,\mathrm{pre}}(i_*G) = G$ is already a sheaf and no sheafification is needed. Recall an open set $W \subseteq S$ is a set of the form $W = U \cap S$ where $U \subseteq X$ is open. We then compute that the unit is given by

$$i^{*,\mathrm{pre}}(i_*G)(W) = \operatorname*{colim}_{V \supseteq W} \Gamma(V; i_*G) = \operatorname*{colim}_{V \supseteq W} \Gamma(V \cap S; G) \to \Gamma(W; G).$$

But by cofinality, we can restrict the colimit to $V \subseteq U$ and it would imply that $V \cap S \subseteq W$ so the filtered diagram takes constant value and the last map is an isomorphism.

Corollary 2.60. *For a continuous map $f : X \to Y$, there is a $*$-adjunction*

$$f^* : \mathrm{Sh}(Y) \rightleftharpoons \mathrm{Sh}(X) : f_*$$

where f_* is constructed in Construction 2.56.

Before we leave this section, we mention a few properties of $\mathrm{Sh}(-)$ that we will be using later. First, "sheaves form a sheaf".

Proposition 2.61. *The presheaf*

$$\begin{aligned} \mathrm{Sh}^* : \mathrm{Op}_X^{\mathrm{op}} &\to \mathrm{Pr}^{\mathrm{L}} \\ U &\mapsto \mathrm{Sh}(U) \\ (V \subseteq U) &\mapsto (\mathrm{Sh}(U) \to \mathrm{Sh}(V)) \end{aligned}$$

is a sheaf in Cat.

Proof. By Theorem 2.53, $\mathrm{Sh}(X)$ is a localization of $\mathrm{PSh}(X)$ with respect to a Grothendieck topology. By [34, Proposition 6.2.2.7], any such localization is a topos and the desired statement is a consequence of (ii) of [34, Theorem 6.1.3.9]. □

Another property is the base-change formula for proper maps. Consider a commuting diagram

$$\begin{array}{ccc} X' & \xrightarrow{q'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{q} & Y \end{array}$$

in topological spaces. The identification $p_*q'_* = q_*p'_*$ induces a map

$$q^*p_* \to q^*p_*q'_*q'^* = q^*q_*p'_*q'^* \to p'_*q'^* \tag{3}$$

between functors from $\mathrm{Sh}(X)$ to $\mathrm{Sh}(Y')$.

Theorem 2.62 ([34, Corollary 7.3.1.18], the nonabelian base change theorem)**.**

$$\begin{array}{ccc} X' & \xrightarrow{q'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{q} & Y \end{array}$$

be a pullback diagram of locally compact Hausdorff spaces and assume p is proper. Then, p_* satisfies base-change, i.e., the canonical map $q^*p_* \to p'_*q'^* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y')$, defined by (3), is an equivalence.

Remark 2.63. What is proven more generally in [34, Section 7.3] is that a proper map $p : X \to Y$ induces a proper morphism $p_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ between topoi. In addition to base-change above, this implies for example that the $*$-pushforward $p_* : \mathrm{Sh}(X; \mathrm{Ani}) \to \mathrm{Sh}(Y)$ preserves filtered colimits. See [34, Remark 7.3.1.5] and [34, Corollary 7.3.4.12].

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