

# Microlocal sheaf theory in noncommutative geometry

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## Abstract

This note is for my talk in the UCSB Seminar on Geometry and Arithmetic on May 11, 2023 and in the Algebraic Geometry seminar in Academia Sinica on May 16th. Beginning with the classical Serre duality in algebraic geometry, we discuss some general ideas in noncommutative geometry, and their appearance in microlocal sheaf theory. The content of the talk is taken from my joint work with Wenyuan Li in [7].

## 1 Commutative and noncommutative geometry

Let  $k$  be a field of characteristic 0 and  $X$  be a proper scheme and  $\omega_X$  be the dualizing sheaf. Then the classical Serre duality asserts that, for  $G, F \in \text{Perf}(X)$ , there is an equivalence

$$\text{Hom}(G, F \otimes \omega_X) = \text{Hom}(F, G)^\vee$$

where the latter is the linear dual. One of the modern interpretation of this equality is that the category  $\text{IndCoh}(X)$  is self-dual, in the following sense:

Let  $(\mathcal{A}, \otimes, 1_{\mathcal{A}})$  be a symmetric monoidal category. An object  $X \in \mathcal{A}$  is said to be dualizable if there exists a triple  $(X^\vee, \eta, \epsilon)$  where  $X^\vee \in \mathcal{A}$  is an object,  $\eta$  and  $\epsilon$  are maps

$$\eta : 1_{\mathcal{A}} \rightarrow X \otimes X^\vee, \quad \epsilon : X^\vee \otimes X \rightarrow 1_{\mathcal{A}}$$

such that the standard triangle equalities  $(\epsilon \otimes \text{id})(\text{id} \otimes \eta) = \text{id}$ ,  $(\text{id} \otimes \epsilon)(\eta \otimes \text{id}) = \text{id}$  hold. The simplest example is when  $\mathcal{A} = \text{Vect}_k$  (resp.  $\text{Vect}_k^\heartsuit$ ) so dualizable objects are  $V \in \text{Perf } k$  (resp. finite dimensional vector spaces), and  $\epsilon$  and  $\eta$  are given by the natural pairing between  $V$  and  $V^\vee$ , which corresponds to the identity under  $V^\vee \otimes_k V = \text{End}(V)$ .

In our case, we take  $\mathcal{A} = \text{Pr}_{st}^L (= \text{Pr}_{st}^L(k))$ , the (very large) category of presentable categories, tensored over  $k$ , whose morphisms are given by  $k$ -linear colimit-preserving functors. In this setting, any compactly generated category  $\mathcal{C} = \text{Ind}(\mathcal{C}_0)$ , of a small stable idempotent complete category  $\mathcal{C}_0$  tensored over  $k$ , is tautologically dualizable with  $\mathcal{C}^\vee = \text{Ind}(\mathcal{C}_0^{op})$ , and  $\epsilon$  and  $\eta$  are both given by the Hom-pairing, which in noncommutative geometry is usually referred as the diagonal bi-module  $\text{Id}_{\mathcal{C}_0} : \mathcal{C}_0^{op} \otimes \mathcal{C}_0 \rightarrow k\text{-Mod}$ .

On the other hand, the category  $\text{IndCoh}(X)$  has an alternative duality data given by

$$\eta = \Delta_*^{\text{Ind}} p^! : k\text{Mod} \rightarrow \text{IndCoh}(X \times X), \quad \epsilon = p_*^{\text{Ind}} \Delta^! : \text{IndCoh}(X \times X) \rightarrow k\text{-Mod}$$

where  $\Delta : X \hookrightarrow X \times X$  is the diagonal and  $p : X \rightarrow \{*\}$  is the projection to a point. Here we also use the fact that  $\text{IndCoh}(X) \otimes \text{IndCoh}(Y) = \text{IndCoh}(X \times Y)$  for any two schemes  $X, Y$ .

Now, the definition of dualizability of  $X \in \mathcal{C}$  implies that  $X^\vee$  is determined by the fact that  $X^\vee \otimes (-)$  is the right adjoint of  $X \otimes (-)$ ; in the case when  $\mathcal{C}$  is closed monoidal, this further implies that  $X^\vee = \underline{\text{Hom}}(X, 1_{\mathcal{C}})$ . The fact that  $\text{IndCoh}(X)^\vee = \text{IndCoh}(X)$ , then implies that, there is an identification  $D_X^{\text{Serre}} : \text{Coh}(X) = \text{Coh}(X)^{op}$  which is compatible with the identifications of the two unit and counit pairs. The classical Serre duality is then a consequence of six-functor yoga. (See the Appendix.)

We point out that, although the statement of classical Serre duality involves the symmetric monoidal product on  $\text{Coh}(X)$ , and hence the reason we refer it as commutative geometry, the existence of the identification  $D_X^{\text{Serre}} : \text{Coh}(X) = \text{Coh}(X)^{op}$  depends only on the symmetric monoidal product on the entire ambient category  $\text{Pr}_{st}^L$ . Thus, when study a general category  $\mathcal{C} \in \text{Pr}_{st}^L$ , one thinks of it as from some sort of noncommutative space.

## 2 Microlocal sheaf theory

We will consider a class of categories in  $\text{Pr}_{st}^L$  of a more topological/symplectic origin and see that a similar phenomenon, that the categorical diagonal bi-module can be realized by a geometrical diagonal. But we first introduce some microlocal sheaf theory. For this part of the talk, one can take the coefficient to be a general symmetric monoidal category  $\mathcal{V}$ , compactly generated by a small stable rigid symmetric monoidal category  $\mathcal{V}_0$ . Some popular choices are  $\mathcal{V} = k\text{-Mod}, \mathbb{Z}\text{-Mod}$ , or  $\text{Sp}$ , the category of spectra.

Let  $M$  be a  $C^\infty$  manifold and we use  $\text{Sh}(M)$  to denote the subcategory of  $\mathcal{V}$ -valued presheaves which turns a Čech colimit diagram of open covers to a limit. The functor  $\text{Sh}$ , really defined for all locally compact Hausdorff spaces, enjoys the usual six-functor formalism. For a sheaf  $F \in \text{Sh}(M)$ , one can assign a set  $\text{SS}(F) \subset T^*M$ , the microsupport of  $F$ , which is a closed, conic, and coisotropic. (The last one is a theorem of Kashiwara and Schapira.) Similar to the support, one way to define it is to say when a point  $(x, \xi)$  is not in  $\text{SS}(F)$ .

**Definition 2.1.** A point  $(x, \xi) \in T^*M$  is not in  $\text{SS}(F)$  if and only if, locally on  $T^*M$ , for any function  $\phi$  such that  $d\phi_x = \xi$ , the object  $(\Gamma_{\{\phi \geq 0\}}(F))_x = 0$ .

Naively, this means that in coordinates, the restriction  $F(U) \rightarrow F(U \cap \{\xi < 0\})$  from a small ball  $U$  to the negative half with respect to  $\xi$  should be an equivalence. We mention that one standard usage of the microsupport is to measure the degree of morphisms being non-isomorphic. For this talk, the main theorem we will use, which connects microlocal sheaf theory to symplectic geometry, is the following theorem of Kashiwara and Schapira [5]:

**Theorem 2.2.** *Let  $M$  be real analytic and assume  $\text{SS}(F)$  is subanalytic. Then  $\text{SS}(F)$  is a singular isotropic if and only if  $F$  is constructible.*

An other important result also by Kashiwara and Schapira, which is later improved by Sheel, John Pardon, and Vivek in [3] is that for any fixed Legendrian  $\Lambda \subseteq S^*M$  satisfying

the above assumption, there is a subanalytic Whitney triangulation  $\mathcal{S}$  such that  $\Lambda \subseteq N_\infty^* \mathcal{S}$ . For a closed subset  $X$  in  $S^*M$ , write

$$\mathrm{Sh}_X(M) := \{F \mid \mathrm{SS}^\infty(F) := ((\mathrm{SS}(F) \setminus 0_M)/\mathbb{R}_{>0}) \subseteq X\},$$

and we mention that, because of the Whitney condition,  $\mathrm{Sh}_{N_\infty^* \mathcal{S}}(M) = \mathrm{Sh}_{\mathcal{S}}(M)$  is equivalent to  $\mathcal{S}\text{-Mod}$ . In other words, one way to understand microsupport is that  $\mathrm{Sh}_\Lambda(M)$  is a subcategory of  $\mathcal{S}\text{-Mod}$ , for some triangulation, with certain arrows being required to be isomorphic.

**Example 2.3.** Consider the case when  $M = S^1$ , the stratification  $\mathcal{S} = \{\{0\}, (0, 1)\}$ , and  $\Lambda = \{(0, -1)\} \subseteq S^*S^1 = S^1 \times \{\pm 1\}$ . The category  $\mathrm{Sh}_{\mathcal{S}}(M)$  is the quiver representation with two vertices and two arrows, going to the same direction, between them. Another name of this category is  $\mathrm{Coh}(\mathbb{P}^1)$ . To be in the subcategory  $\mathrm{Sh}_\Lambda(M)$ , the requirement is for one of the arrows, a fixed one, to be isomorphic. Again, this category has another name which is  $\mathrm{Coh}(\mathbb{A}^1)$  and they are the simplest examples of toric mirror symmetry.

Now, before going back to noncommutative geometry, there is a one more thing we have to mention, i.e., for any closed conic subset  $X \subseteq T^*M$ , the inclusion  $\mathrm{Sh}_X(M) \hookrightarrow \mathrm{Sh}(M)$  have both left and right adjoints. Combining with the fact that  $\mathrm{Sh}_\Lambda(M) \subseteq \mathcal{S}\text{-Mod}$  for some triangulation  $\mathcal{S}$ , we conclude that  $\mathrm{Sh}_\Lambda(M)$  is compactly generated and we will, consequently denote its compact objects by  $\mathrm{Sh}_\Lambda(M)^c$ . We mention that it contains  $\mathrm{Sh}_\Lambda(M)_0^b$  where the superscript ‘b’ means perfect stalks and the subscript ‘0’ means compact support.

### 3 Dualizability of sheaves with prescribed Legendrian microsupport

The main observation of this talk is that, for a Legendrian  $\Lambda \subseteq S^*M$ , the dual of the category  $\mathrm{Sh}_\Lambda(M)$  admits a description  $\mathrm{Sh}_\Lambda(M)^\vee = \mathrm{Sh}_{-\Lambda}(M)$  where  $-\Lambda \subseteq S^*M$ , is the image of  $\Lambda$  under the involution  $(x, [\xi]) \mapsto (x, [-\xi])$ . Similar to the algebraic geometry case, the idea is to use the geometrical diagonal to represent the categorical diagonal. Again, we will utilize the identification  $\mathrm{Sh}_{-\Lambda}(M) \otimes \mathrm{Sh}_\Lambda(M) = \mathrm{Sh}_{\Lambda \times -\Lambda}(M \times M)$ , where for Legendrians  $\Lambda \subseteq S^*M$ ,  $\Sigma \subseteq S^*N$  at infinite, we use the notation  $\Lambda \times \Sigma$  to mean the Legendrian

$$((\mathbb{R}_{>0}\Lambda \cup 0_M) \times (\mathbb{R}_{>0}\Sigma \cup 0_N)) \setminus 0_{M \times N}/\mathbb{R}_{>0}.$$

One problem, however, is that  $\mathrm{SS}(1_\Delta) = N^*\Delta$  is never going to have the correct microsupport condition. The solution is very simple: One simply applies the left adjoint,  $\iota_{\Lambda \times -\Lambda}^*$ , of the inclusion  $\mathrm{Sh}_{\Lambda \times -\Lambda}(M \times M) \hookrightarrow \mathrm{Sh}(M \times M)$  to force the microsupport condition. Then a similar computation as the algebraic geometry case provides the following:

**Theorem 3.1** ([7, Theorem 1.16]). *The triple  $(\mathrm{Sh}_{-\Lambda}(M), \eta, \epsilon)$  where  $\eta$  and  $\epsilon$  are given by*

$$\eta = \iota_{\Lambda \times -\Lambda}^* \Delta_* p^* : k \text{ Mod} \rightarrow \mathrm{Sh}_{\Lambda \times -\Lambda}(M \times M),$$

and

$$\epsilon = p_! \Delta^* : \mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M) \rightarrow k\text{-Mod}$$

exhibit  $\mathrm{Sh}_{-\Lambda}(M)$  as a dual of  $\mathrm{Sh}_\Lambda(M)$ .

Because of the above theorem, there exists an identification  $D_\Lambda : \text{Sh}_\Lambda(M)^{c,op} = \text{Sh}_{-\Lambda}(M)^c$ . We explain that this identification gives something familiar for those  $F$  in  $\text{Sh}_\Lambda(M)_0^b$ . Uniqueness of the counit  $\epsilon$  implies that for  $F \in \text{Sh}_\Lambda(M)^c$  and  $H \in \text{Sh}_{-\Lambda}(M)^c$ , there is an identification

$$\text{Hom}(D_\Lambda(F), H) = p_!(F \otimes H).$$

Assume  $F$  in  $\text{Sh}_\Lambda(M)_0^b$  and one can perform the computation

$$p_!(F \otimes H) = p_*(F \otimes H) = p_*((F^\vee)^\vee \otimes H) = p_*\Delta^*(\mathcal{H}\text{om}(\pi_2^*F^\vee, \pi_1^*H)).$$

We would like to give an explanation of the last term but we need to take a detour and explain what isotopies of sheaves are.

Recall that a contact manifold  $(V^{2n-1}, \xi)$  is an odd dimensional manifold with a hyperplane distribution  $\xi \subseteq TV$  which is maximally nonintegrable, meaning that, if one takes any local expression  $\xi = \ker \alpha$ , then  $\alpha \wedge (d\alpha)^{n-1}$  is nonvanishing. A contact isotopy  $\varphi_t : V \rightarrow V$  is a smooth family of diffeomorphism such that  $\varphi_0 = \text{id}$  and  $(\varphi_t)_*\xi = \xi$ . One key theorem which introduces contact geometry into microlocal sheaf theory is the Guillermou-Kashiwara-Schapira sheaf quantization:

**Theorem 3.2** ([4]). *Let  $\Phi : S^*M \times I \rightarrow S^*M$  be a contact isotopy. Then there exists a unique sheaf  $K(\Phi) \in \text{Sh}(M \times M \times I)$  such that  $K(\Phi)_{t=0} = 1_\Delta$  and  $\text{SS}^\infty(K(\Phi))$  is contained in the movie of  $\Phi$ .*

Let  $q_x : M_x \times M_y \times I_t \rightarrow M_x$  and  $q_{yt} : M_x \times M_y \times I_t \rightarrow M_y \times I_t$  be the corresponding projections. Then for a sheaf  $F \in \text{Sh}(M)$ , the convolution  $K(\Phi) \circ F := q_{yt_!}(K(\Phi) \otimes q_x^*F)$  is an object in  $\text{Sh}(M \times I)$ , which can be thought of as an  $I$ -family sheaves. For a fixed  $t \in I$ , denote by  $F_t$  the restriction  $(K(\Phi) \circ F)|_t = K(\Phi)|_t \circ F$ . The main property of  $F_t$  is that  $\text{SS}^\infty(F_t) = \varphi_t(\text{SS}^\infty(F))$  and we thus think of  $K(\Phi) \circ F$  as isotopies of  $F$ , which sometimes people like to refer this as a wrapping.

Note that  $S^*M$  is co-orientable, i.e., there is a global one form  $\alpha$  such that  $\ker \alpha$  gives the contact structure. In this case, one say an isotopy is positively, which really should be called non-negative, if  $\alpha(\dot{\varphi}_t) \geq 0$ . When  $\Phi$  is positively, an extra property of  $K(\Phi)$  is that, for  $s \leq t$ , there is a continuation map  $K(\Phi)|_s \rightarrow K(\Phi)|_t$ .

**Proposition 3.3** ([7, Proposition 4.3]). *Let  $F, H \in \text{Sh}_\Lambda(M)_0$  and denote by  $G^w$  a non-specified small positive push-off displacing  $\Lambda$  from itself. Then there is an identification*

$$\Gamma(M; \Delta^* \mathcal{H}\text{om}(\pi_2^*F, \pi_1^*H)) \xrightarrow{\sim} \text{Hom}(F^w, H).$$

To get from  $F^w$  back to  $\text{Sh}_\Lambda(M)$ , we mention that the adjoints of the inclusion  $\text{Sh}_X(M) \hookrightarrow \text{Sh}(M)$ , which we mentioned that it exists, also admit a contact geometrical description.

**Theorem 3.4** ([6, Theorem 1.3]). *The left (resp. right) adjoint of the inclusion  $\text{Sh}_X(M) \hookrightarrow \text{Sh}(M)$  is given by*

$$\mathfrak{W}_X^+(F) := \text{colim}_{w:X^c} F^w \quad (\text{resp. } \mathfrak{W}_X^-(F) := \lim_{w^-:X^c} F^{w^-})$$

where  $w$  (resp.  $w^-$ ) runs over all positive (resp. negative) wrapping compactly supported away  $X$ .

Thus, combining all the above identifications, we obtain that

**Corollary 3.5.** *For  $F \subseteq \mathrm{Sh}_\Lambda(M)_0^b$ , the “Serre dual” of  $F$  with respect to  $\Lambda$  is given by*

$$D_\Lambda(F) = \mathfrak{W}_\Lambda^+((F^\vee)^w) \in \mathrm{Sh}_{-\Lambda}(M)^c$$

where  $F^\vee := \mathcal{H}\mathrm{om}(F, 1_M)$  is the naive dual of  $F$ .

## 4 Spherical functors

We give a noncommutative geometric explanation of the endofunctor on  $\mathrm{Sh}_\Lambda(M)$ , sending  $F$  to  $\mathfrak{W}_\Lambda^+(F^w)$  where we again fix a small wrapping  $w$  displacing  $\Lambda$  from itself.

**Definition 4.1.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor with left and right adjoints. Then, one obtains two adjunction pairs of endofunctors  $S^\pm$  on  $\mathcal{C}$  and  $T^\pm$  on  $\mathcal{D}$ , which are given by the following four fiber sequences:

$$\begin{aligned} F^L F &\rightarrow \mathrm{id}_{\mathcal{C}} \rightarrow S^+, & T^+ &\rightarrow \mathrm{id}_{\mathcal{D}} \rightarrow F F^L, \\ S^- &\rightarrow \mathrm{id}_{\mathcal{C}} \rightarrow F^R F, & F F^R &\rightarrow \mathrm{id}_{\mathcal{D}} \rightarrow T^-. \end{aligned}$$

When all four of  $S^\pm$  and  $T^\pm$  are equivalences, we say that  $F$  is a spherical functor.

In our case, the functor  $F$  comes from microlocalization: One can define the category-valued sheaf  $\mu\mathrm{sh}$  by sheafifying the presheaf

$$\Omega \mapsto \mathrm{Sh}(M) / \mathrm{Sh}_{\Omega^c}(M)$$

where  $\Omega \subseteq T^*M$  runs over conic open sets in the cotangent bundle. The notion of microsupport descends to  $\mu\mathrm{sh}$  and, for  $F \in \mu\mathrm{sh}(\Omega)$ , the microsupport on  $\Omega$ ,  $\mathrm{SS}_\Omega(F) \subseteq \Omega$ , is well-defined.

**Definition 4.2.** We denote by  $\mu\mathrm{sh}_\Lambda$  the subsheaf with objects microsupported in  $\Lambda$ .

An alternative construction of  $\mu\mathrm{sh}_\Lambda$  is by sheafifying the presheaf  $\Omega \mapsto \mathrm{Sh}_\Lambda(M) / \mathrm{Sh}_{\Lambda \setminus \Omega}(M)$  and in this case, one sees that there is a canonical functor  $m_\Lambda : \mathrm{Sh}_\Lambda(M) \rightarrow \mu\mathrm{sh}_\Lambda(\Lambda)$  coming from the inclusion  $\dot{T}^*M \subseteq T^*M$ . For tautological reason, the functor  $m_\Lambda$  has both left and right adjoint and we use  $S_\Lambda^\pm$  and  $T_\Lambda^\pm$  to denote the corresponding functors mentioned above.

**Theorem 4.3.** *The functor  $S_\Lambda^+ : \mathrm{Sh}_\Lambda(M) \rightarrow \mathrm{Sh}_\Lambda(M)$  can be computed as*

$$S_\Lambda^+(F) = \mathfrak{W}_\Lambda^+(F^w)$$

where  $w$  is any small positive wrapping, displacing  $\Lambda$  from itself. When  $\Lambda \subseteq S^*M$  is swappable, i.e., when there exists a small neighborhood  $\mathcal{U}$  of  $\Lambda$  such that there exists a positive isotopy sending  $\mathcal{U}^w$  to  $\mathcal{U}^{w^-}$ , then  $S_\Lambda^+$  is an equivalence. A similar statement holds for  $S_\Lambda^-$ .

*Proof.* The statement holds locally, which can alternatively be viewed as the definition of  $\mu\mathrm{sh}_\Lambda$  as in [8], and one argues that such picture glues globally.  $\square$

**Corollary 4.4** ([7, Proposition 5.28]). *For a swappable  $\Lambda \subseteq S^*M$ , the Serre functor on  $\mathrm{Sh}_\Lambda(M)_0^b$  is given by*

$$F \mapsto S_\Lambda^-(F \otimes \omega_M).$$

## A Some derived algebraic geometry

The main goal of this section is to explain how one can get the classical Serre duality from  $\text{IndCoh}(X)^\vee = \text{IndCoh}(X)$ . For this talk, we will assume our scheme to be almost finite type [2, 1.7.2, 3.5.1 of I.2]. That is, they are separable quasi-compact scheme, which are convergent as prestacks, and any truncation is of finite type. For such a scheme  $X$ , the subcategory  $\text{Perf}(X)$  consisting of dualizable objects will be the same as  $\text{QCoh}(X)^c$ , compact objects in  $\text{QCoh}(X)$ . In particular, taking global sections  $\Gamma(X; -)$  preserves colimit.

Another small category one like to consider is  $\text{Coh}(X)$ , the subcategory of  $\text{QCoh}(X)$  consisting of bounded chain complexes with coherent cohomology [2, 1.1.1., II.1]. The Ind-coherent sheaves are objects of the category  $\text{IndCoh}(X) := \text{Ind}(\text{Coh}(X))$ , which admits a functor  $\Psi_X : \text{IndCoh}(X) \rightarrow \text{QCoh}(X)$  [2, 1.1.2, II.1]. When assuming the scheme  $X$  to be eventually connective, i.e., when  $\mathcal{O}_X \in \text{Coh}(X)$  and the inclusion  $\text{Perf}(X) \subseteq \text{Coh}(X)$ , which is usually a proper inclusion as shown in the following example, gives a left adjoint  $\Theta_X : \text{QCoh}(X) \rightarrow \text{IndCoh}(X)$  of  $\Psi_X$ .

**Example A.1.** Let  $X = \text{Spec}(k[t]/(t^2))$ . Then  $k \in \text{Coh}(X) \setminus \text{Perf}(X)$  and the key observation is that  $k$  is the same as the unbounded chain complex

$$\cdots \xrightarrow{t} k[t]/(t^2) \xrightarrow{t} k[t]/(t^2) \xrightarrow{t} k[t]/(t^2) \rightarrow 0.$$

Now we note that tensor products on  $\text{QCoh}(X)$  induces an  $\text{Perf}(X)$ -action on  $\text{Coh}(X)$ , and we obtain an  $\text{QCoh}(X)$ -action on  $\text{IndCoh}(X)$ ,

$$\text{QCoh}(X) \otimes \text{IndCoh}(X) \rightarrow \text{IndCoh}(X),$$

by taking Ind-completion. For a  $F \in \text{IndCoh}(X)$ ,  $(-) \otimes F$  has a right adjoint given by  $\underline{\text{Hom}}(F, -)$  [1, 9.5.2]. For a scheme  $X$ , the dualizing sheaf  $\omega_X := p^!k$ , where  $p : X \rightarrow \text{Spec } k$  is the projection to a point, is defined to be the  $!$ -pullback of the base field  $k$ . One can show that  $D_X^{\text{Serre}}(F) : \text{Coh}(X)^{op} = \text{Coh}(X)$  is given by  $D_X^{\text{Serre}}(F) = \underline{\text{Hom}}(F, \omega_X)$  [1, Proposition 9.5.7] by comparing the two counits.

With the above background, we can deduce the classical statement of Serre duality, that on a proper scheme  $X$ , there is an equivalence,

$$\text{Hom}(G, F \otimes \omega_X) = \text{Hom}(F, G)^\vee$$

for any  $F, G \in \text{Perf}(X)$ , with the following computation:

We first notice that when  $X$  is proper,  $p^! : k\text{-Mod} \rightarrow \text{IndCoh}(X)$  is the right adjoint of  $p_*^{\text{Ind}} : \text{IndCoh}(X) \rightarrow k\text{-Mod}$ . (Recall that, in general,  $p^!$  is neither a left adjoint nor a right adjoint.) Thus, the right hand side is given by

$$\text{Hom}(F, G)^\vee = \text{Hom}_k(\text{Hom}_X(F, G), k) = \text{Hom}_X(\mathcal{H}\text{om}(F, G), \omega_X).$$

Here, we use the fact that  $\mathcal{H}\text{om}(F, G) \in \text{Perf}(X)$ , so  $\text{Hom}_X(F, G) = p_*^{\text{Ind}} \mathcal{H}\text{om}(F, G)$ . By definition,  $\text{Perf}(X)$  consists of dualizable objects so the internal Hom  $\mathcal{H}\text{om}(F, G) = F^\vee \otimes E$  is given by a tensor, i.e.,

$$\text{Hom}(F, G)^* = \text{Hom}_X(F^\vee \otimes G, \omega_X) = \text{Hom}_X(G, \mathcal{H}\text{om}(F^\vee, \omega_X)) = \text{Hom}_X(G, F \otimes \omega_X).$$

## References

- [1] Dennis Gaitsgory. Ind-coherent sheaves. *arXiv:1105.4857v7*, 2012.
- [2] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.
- [3] Sheel Ganatra, John Pardon, and Vivek Shende. Covariantly functorial wrapped Floer theory on Liouville sectors. *Publ. Math. Inst. Hautes Études Sci.*, 131:73–200, 2020.
- [4] Stéphane Guillermou, Masaki Kashiwara, and Pierre Schapira. Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems. *Duke Math. J.*, 161(2):201–245, 2012.
- [5] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1994. With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original.
- [6] Christopher Kuo. Wrapped sheaves. *arXiv:2102.06791*, 2021.
- [7] Christopher Kuo and Wenyuan Li. Duality and spherical adjunction from microlocalization – an approach by contact isotopies. *arXiv:2210.06643*, 2022.
- [8] David Nadler. Wrapped microlocal sheaves on pairs of pants. *arXiv:1604.00114*, 2016.