Koszul duality for algebras

Christopher Kuo

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Abstract

Let A be an algebra. The Koszul duality is a type of derived equivalence between modules over A and modules over its Koszul dual $A^!$. In this talk, we will talk about the general framework and then focus on the classical cases as well as examples.

1 Introduction

A standard way to obtain equivalence between categories of modules is through Morita theory. Let $\mathcal{C}$ be a representable, k-linear, cocomplete category. Let $X \in \mathcal{C}$ be a compact generator which means the functor $\mathrm{Hom}_{\mathcal{C}}(X, \cdot)$ is cocontinuous and conservative. Let $A = \mathrm{End}_{\mathcal{C}}(X)^{op}$ be the opposite algebra of the endomorphism algebra of X. Then we have the Morita equivalence, $X \cong A{-}\mathrm{Mod}$ which is given by the assignment $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Let k be a field such that $\mathrm{char}(k) \neq 2$ and A be an algebra over k. Following the above framework, one might hope that in some good cases there's an equivalence $A - \mathrm{Mod} \cong \mathrm{Hom}_A(k, k)^{op} - \mathrm{Mod}$. Unfortunately, this is not true in general. Consider the case $A = k[x]$ and equip k with the trivial A-module structure. There is a two term projective resolution of k by $0 \to k[x] \xrightarrow{x} k[x] \to k \to 0$ and $\mathrm{Hom}_A(k, k) \cong k[\epsilon]$ with $\epsilon^2 = 0$. So we can ask if that the assignment $M \mapsto \mathrm{Hom}_{k[x]}(k, M)$ induces a equivalence $k[x] - \mathrm{Mod} \cong k[\epsilon] - \mathrm{Mod}$?

The answer is false. For example, the functor $\mathrm{Hom}_{k[x]}(k, \cdot)$ kills non-zero objects. Let $k_a = k[x]/(x - a)$. They are modules with underlining vector space k such that $x \cdot 1 = a$. The image $\mathrm{Hom}_{k[x]}(k, k_a) \cong 0 \to k \xrightarrow{a} k \to 0$ is isomorphic to the 0 object in the derived category when $a \neq 0$. There is also some completeness issue which we will talk about later.

The goal of this talk is to explore the various settings that make this heuristic work and to provide examples of the duality.

2 The graded version

In this notes, we abuse the notation and write $\otimes_k$ by just $\otimes$.

Definition 2.1. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded algebra such that $A_0 = k$ and $\dim_k A_i < \infty$. We say A is Koszul if there is a graded projective resolution of the trivial module k,

$$\cdots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow k \longrightarrow 0$$

such that $P^i = AP^i_i$ is generated at $\deg i$. In this case, we define the Koszul dual algebra $A^!$ to be the extension algebra $\mathrm{Ext}^{\cdot}_A(k, k)^{op}$ where non-graded version of Ext.

Example 2.1. Let V be a finite dimension k vector space and $A = S(V)$ its symmetric algebra. The algebra A is Koszul since k is resolved by

$$\cdots \longrightarrow S(V) \otimes \Lambda^2 V \longrightarrow S(V) \otimes V \longrightarrow S(V) \longrightarrow k \longrightarrow 0,$$

where the differential is given by

$$f(x_1, \cdots, x_n) v_{i_1} \wedge \cdots \wedge v_{i_j} \mapsto \sum_k (-1)^{1+k} f(x_1, \cdots, x_n) x_{i_k} v_{i_1} \wedge \cdots \wedge \widehat{v_{i_k}} \wedge \cdots \wedge v_{i_j}.$$

Fact 2.1. *If A is Koszul, then $A^!$ is also Koszul. Moreover, $(A^!)^! \cong A^!$.*

When A is Koszul, A can be presented by generators at deg 1 and relations at deg 2.

Definition 2.2. An algebra A is called quadratic if $A \cong T(V)/R$ for some finite dimensional k-vector space V and subspace $R \subseteq V^{\otimes 2}$.

Fact 2.2. *If A is Koszul, then A is quadratic. Write $A = T(V)/R$ for some V and R. Then $A^! \cong T(V^*)/R^\perp$.*

Example 2.2. Let V be a finite dimensional vector space. Set $R = \{0\}$, then $A = T(V)$ the tensor algebra. In this case, $R^\perp = V^{\otimes 2}$ and $A^! = k \oplus V^*$ with the trivial algebra structure. Note A and $A^!$ are Koszul since it the trivial A-module k can be resolved by the two term resolution,

$$0 \to T(V) \otimes V \to T(V) \to k \to 0.$$

Example 2.3. Let V be a finite dimensional vector space. Set $R = \langle xy - yx \rangle_k$, then $A = S(V)$ the symmetric algebra of V. In this case, $R^\perp = \langle xy + yx \rangle$ and $A^! = \Lambda^\cdot V^*$ the exterior algebra of V^*. As mentioned before, A and $A^!$ are Koszul.

For a quadratic algebra $A = T(V)/R$, we use the same notation $A^!$ to denote $T(V^*)/R^\perp$ and call it the Koszul dual algebra of A. Then for any quadratic algebra, there is the Koszul complexes

$$\cdots \longrightarrow A \otimes (A^!_2)^* \longrightarrow A \otimes (A^!_1)^* \longrightarrow A \longrightarrow k \longrightarrow 0$$

where the differential is given by $(df)(a) = \sum_\alpha f(a\hat{v}_\alpha) v_\alpha$. Here we identify $A \otimes (A^!_i)^*$ with $\mathrm{Hom}_k(A^!_i, A)$. The finite set $\{v_\alpha\}$ is any basis of V and $\{\hat{v}_\alpha\}$ is the dual basis. Recall that $\sum_\alpha \hat{v}_\alpha \otimes v_\alpha$ is the coevaluation in $V^* \otimes V \cong \mathrm{Hom}_k(V, V)$.

Fact 2.3. *When A is Koszul, the Koszul complexes gives a graded projective resolution of k.*

Now following the Morita framework, we want to define an adjunction pair $F : K(A^!) \rightleftarrows K(A) : G$ by a Hom/$\otimes$ adjunction. Here $K(A)$ is the homotopy category of the category of chain complexes of A-modules. Let $T = A \otimes A^!$ which is a $A - A^!$ bimodule. There is a natural differential structure on T which is given by

$$u \otimes a \mapsto \sum_\alpha u x_\alpha \otimes \check{x}_\alpha a.$$

[0]The functors F and G go in the opposite direction in [1] and [4]

We define $F(N) = T \otimes_{A^!} N$ and $G(M) = \mathrm{Hom}_A(T, M)$. Here Hom_A and $\otimes_{A^!}$ means the Hom and tensor chain complexes. More explicitly, identify $F(N) = T \otimes_{A^!} N = A \otimes A^! \otimes_{A^!} N \cong A \otimes N$. The degree p piece $F(M)^p$ is $A^! \otimes M^p$ and the differential is given by

$$d_{F(N)}(a \otimes u) = \sum_{\alpha} a x_\alpha \otimes \check{x}_\alpha m + a \otimes d_N(n).$$

The complex $G(M)$ can be identified as $\mathrm{Hom}_k(A^!, M)$ and the chain complex structure can be described in a similar wau. In the case when N and M are graded, we can equip $F(N)$ and $G(M)$ with gradings.

The theorem is the following.

Theorem 2.1 (Beilinson-Ginzburg-Soergel([1]). *The pair of functors $F : K(A^!) \rightleftarrows K(A) : G$ forms an adjunction. When restricts to one side bounded chain complexes of graded modules the adjunction $F : D^-(\text{graded } A^!) \rightleftarrows D^+(\text{graded } A) : G$ induce an equivalence of categories. Here D^- means bounded above derived category and similar for D^+.*

A natural question is whether we can extend this equivalence to unbounded chain complexes $F : D(\text{graded } A^!) \rightleftarrows D(\text{graded } A) : G$?

The answer is false. Consider $A = k[x]$ with the natural grading for example. It is a compact object in $D(A)$ and $G(A) \cong 0 \to k[x] \xrightarrow{x} k[x] \to 0 \cong k\langle 1\rangle [-1]$. The latter is not compact in $D(A^!) = D(k[\epsilon])$. Resolve k by

$$\longrightarrow k[\epsilon]\langle 2\rangle \xrightarrow{\epsilon} k[\epsilon]\langle 1\rangle \xrightarrow{\epsilon} k[\epsilon] \to k \to 0.$$

From this expression, we can see $\mathrm{Hom}_{D(k[\epsilon])}(k, \bigoplus_{n\in\mathbb{N}} k\langle n\rangle [n]) \neq \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}_{D(k[\epsilon])}(k, k\langle n\rangle [n])$.

3 A more general setup

Note that $FG(M)$ is the Koszul complex

$$\cdots \to A \otimes (A^!_p)^* \otimes M \to A \otimes (A^!_{p-1})^* \otimes M \to A \otimes M.$$

So the canonical maps induce by the adjunction $FG(M) \to M$ and $N \to GF(N)$ are quasi-isomorphisms.

We denote the full subcategory of acyclic chain complexes in $K(A)$ by Z_A. Define $N_A \coloneqq \{N \in Z_A \big| G(N) \in Z_{A^!}\}$ and define $N_{A^!}$ in a similar fashion. Then formal homological algebra arguments imply:

Theorem 3.1 (Fløstard ([4]). *The adjunction pairs $F : K(A^!) \rightleftarrows K(A) : G$ descends to an equivalence of categories $\overline{F} : K(A^!)/N_{A^!} \rightleftarrows K(A)/N_A : \overline{G}$.*

Remark 3.1. Note that we don't need to consider graded modules in this setting. Moreover, $0 \to k \xrightarrow{a} k \to 0$ is no longer 0 for the case $A = k[x]$.

Another advantage of not using graded modules is that we can consider a wider varieties of algebras.

Definition 3.1. An algebra U is called a quadratic-linear-scalar algebra if $U \cong T(V)/P$ for some finite dimensional k-vector space V and subspace $P \subseteq k \oplus V \oplus V^{\otimes 2}$ such that $P \cap (k \oplus V) = \{0\}$. Let $R = p_2(P)$ be the deg 2 projection of P. We call $A = T(V)/R$ the associated quadratic algebra of U.

Note that the assumption $P \cap (k \oplus V) = \{0\}$ implies there are k-linear maps $\alpha : R \longrightarrow V$ and $\beta : R \longrightarrow k$ such that $P = \{x + \alpha(x) + \beta(x) \big| x \in R\}$ is a graph.

Proposition 3.1. *Assume A is Koszul. Then U is of PBW-type, i.e., the canonical map $A \twoheadrightarrow grU$ is an isomorphism if and only if $\alpha^* : V^* \cong A^!_1 \to A^!_2 \cong R^*$ can be extended to an (anti)-derivation d such that if we set $c = \beta^*(1)$ such that $d(c) = 0$ and $d^2(b) = [c, b]$.*

In general, a triple (B, d, c) satisfying the above condition is called a curved differential graded algebra.

Example 3.1. Let $V = (\mathfrak{g}, [\cdot,\cdot])$ be a finite dimensional Lie algebra. Set $P = \langle xy - yx - [x, y]\rangle_k$, then $U = U(\mathfrak{g})$, the universal enveloping algebra. In this case, $R = \langle xy - yx\rangle_k$ so $R^\perp = \langle xy + yx\rangle_k$ and $A^! = \Lambda^{\boldsymbol{\cdot}}\mathfrak{g}^*$. The differential is extended from the linear map $d : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ which is given by $(d\lambda)(x, y) = \lambda([x, y])$. The resulting differential graded algebra $(\Lambda^{\boldsymbol{\cdot}}\mathfrak{g}^*, d)$ is the Chevalley-Eilenberg cochain complex.

Example 3.2. Let (V, w) be a finite dimensional vector space equipped with a non-degenerate bilinear form w. Set $P = \langle x - w(x)\rangle_k$, then $U = Cl(V, w)$, the Clifford algebra on V associated with w. In this case, $R = \langle xy + yx\rangle_k$ so $R^\perp = \langle xy - yx\rangle_k$ and $A^! = S(V^*)$. The curvature element is the quadratic function $w(x)$.

Most of the constructions can be generalized to this setting. Let $K(U)$ be the homotopy category of chain complexes of U-modules, $K(A^!, d, c)$ be the homotopy category of curved differential graded modules. The latter consists of objects M such that $d_M(am) = d_{A^!}(a)m + (-1)^{|a|}ad_M(m)$, $d^2_M(m) = cm$ for $a \in A^!$ and $m \in M$. Also, set $T = U \otimes A^!$ with the right cdg module structure $u \otimes a \mapsto \sum_\alpha ux_\alpha \otimes \check{x}_\alpha a + u \otimes d_{A^!}(a)$. The construction goes parallel to the previous case.

Theorem 3.2 (Fløystad [4])**.** *The pair $F : K(A^!, d, c) \rightleftarrows K(U) : G$ still forms an adjunction. When U is Quadratic-linear, i.e., the curvature $c = 0$, the adjunction descends to $\overline{F} : K(A^!, d)/N_{(A^!,d)} \rightleftarrows K(U)/N_U : \overline{G}$ which gives an equivalence of category.*

Remark 3.2. The case $U = U(\mathfrak{g})$ is, in some sense, the historical inspiration for Koszul duality. Recall for a topological space X, one can construct a Sullivan minimal model (M_X, d) which is a dga such that $(M(X), d) \xrightarrow{qis} C^*(X, \mathbb{Q})$. There is also the Lie model $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$ with the Whitehead bracket being its Lie bracket such that $UL_X \xrightarrow{\cong} H_*(\Omega X; \mathbb{Q})$. Roughy speaking, an homotopy (L_∞) version of $(UL_X)^! = (M(X), d)$ should hold. See [12] for details.

4 Clifford modules and matrix factorization

It's harder to describe which kind of morphisms we should localize when there is non-trivial curvature. So the speaker decided not to talk about the general case but just the case when $U = Cl(V, w)$ the Clifford algebras. In this case, the dual objects will be the matrix factorization.

Definition 4.1. Let $R = k[x_1, \cdots, x_r]$ be the polynomial ring with r variables and fix a polynomial $w(x)$. A matrix factorization of $w(x)$ is a sequence $M_0 \xrightarrow{\phi} M_1 \xrightarrow{\psi} M_0$ which is isomorphic to a sequence of the form $R^n \xrightarrow{P} R^n \xrightarrow{Q} R^n$ such that $P, Q \in M_{n\times n}(R)$ and $PQ = QP = w(x)\,\mathrm{id}$. We denote this by $\mathbf{M} = (P, Q)$.

A morphism between $\mathbf{M} = (P, Q)$ and $\mathbf{M}' = (P', Q')$ will be a diagram which is isomorphic to

$$\begin{tikzcd}
R^n \arrow[r, "P"] \arrow[d, "A"'] & R^n \arrow[r, "Q"] \arrow[d, "D"'] & R^n \arrow[d, "A"'] \\
R^n \arrow[r, "P'"] & R^n \arrow[r, "Q'"] & R^n
\end{tikzcd}$$

We denote the category of matrix factorizations of $w(x)$ by $MF(w)$.

Fix a non-degenerate quadratic form $w(x)$. We can define a pair of functors $H : MF(w) \rightleftarrows Cl\,\mathrm{Mod}(V, w) : M$ in an explicit way. The right hand side is the category of Clifford modules.

For $M = (P, Q) \in MF(w)$, write $w(x) = \sum w_{ij} x_i x_j$. Define the matrix Q_i by

$$Q_i = \begin{bmatrix} 0 & \frac{\partial Q}{\partial x_i}(0) \\ \frac{\partial P}{\partial x_i}(0) & 0 \end{bmatrix}.$$

The matrices Q_i's satisfy the equation $Q_i Q_j + Q_j Q_i = 2w_{ij}$ so we can use them to induce a Clifford module structure on $k^n \oplus k^n$. We denote this Clifford module by $H(M)$.

For a Clifford module $A = A_0 \oplus A_1$, we define a matrix factorization $M(A)$ by the sequence $k[V^*] \otimes A_0 \xrightarrow{\phi} k[V^*] \otimes A_1 \xrightarrow{\psi} k[V^*] \otimes A_0$ where $\phi(1 \otimes a) = \sum_i x_i \otimes e_i a$ and similarly for ψ.

Theorem 4.1 (Bertin [2])**.** *The pair $H : MF(w) \rightleftarrows K(f.d.CL(V, w)) : M$ forms an equivalence of category.*

5 Other Settings

There are a few more ways to solve the completion issue. For example, Keller in his notes ([7]) uses a even more general setting. Instead of algebras, we can consider differential graded algebras without any finiteness condition. In this setting, the dual objects will be the differential coalgebras. So instead of considering $\mathrm{Ext}^{\bullet}_A(k, k)$, the main player will be a dg coalgebra $\mathrm{Tor}^{\bullet}_A(k, k)$. At the end, there is an equivalence of categories between the (unbounded) derived category of dg A-modules and the coderived category of dg comodules over the Koszul dual of A.

We don't need any grading in this setting and $k[x]$ will be regarded as a dga concentrated in (differential) deg 0.

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