

Hochschild-Kostant-Rosenberg theorem through loop spaces

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Abstract

Let $X = \text{Spec } A$ be a affine scheme. The Hochschild-Kostant-Rosenberg theorem relates the Hochschild homology of X with the differential forms on X :

$$\text{HH}(\text{QCoh}(X)) = \Lambda_X^\bullet(\Omega(X)).$$

In this two-part talk, we recall the setting of derived algebraic geometry and introduce some standard constructions. Then we will see that once everything is set up, we can provide a quick geometric proof and extend the statement to derived scheme X :

$$\mathcal{O}_{\mathcal{L}X} \cong \Omega_X^\bullet$$

as \mathcal{O}_X -algebras where $\Omega_X^\bullet = \text{Sym}^\bullet(T^*X[1])$ is the symmetric algebra on the shifted cotangent complex.

We mostly follow Brav's talk ([Bra18]) and Ben-Zvi and Nadler's paper ([BD10]) in this talk. The foundational material can be found in Gaitsgory and Rozenblyum's book ([GR16]). Any errors introduced are mine.

1 Settings

Through this talk, we fix a ground field k of characteristic 0 and fix a model for the ∞ -category of ∞ -groupoids \mathcal{S} . Unless specified, all the categorical terminologies are understood in the infinit/derived categorical sense.

Definition 1.1. By a derived k -algebra A , we mean a coconnective commutative dg k -algebra meaning $H^i(A) = 0$ for all $i > 0$. We write DGA^- for the category of derived k -algebras. We denote Sch^{aff} the opposite category $(DGA^-)^{op}$ and refer the objects of this category as affine schemes.

Remark 1.2. One can also use simplicial commutative rings or E_∞ rings as the model of affine schemes. All three of them give an equivalent theory. However, in positive characteristic, dg algebras are ill-behaved and commutative rings and E_∞ rings give different (well-behaved) theories.

In this talk, we will take the functor of points approach and the basic geometric objects are prestacks.

Definition 1.3. A prestack over k is a functor

$$X : (\text{Sch}^{\text{aff}})^{\text{op}} \rightarrow \mathcal{S}$$

from the category of affine schemes to groupoids.

We can equip several different Grothendieck topologies on the category of prestacks such as flat, ppf, étale or Zariski topology.

Definition 1.4. A prestack X is a stack if it satisfies étale descent and X is a scheme if it's a stack with affine diagonal and has a Zariski atlas. We denote the category of stacks by DSt and the category of schemes by Sch .

Remark 1.5. People usually refer this type of schemes as separated schemes. We assume this condition to avoid pathological cases. Replacing the condition of satisfying Zariski descent by étale descent gives the same objects.

Now we want to define quasi-coherent sheaves. For an affine scheme $\text{Spec } A$, we define the category of quasi-coherent sheaves $\text{QCoh}(\text{Spec } A)$ to be $A\text{-Mod}$. If there is morphism $f : A \rightarrow B$, then B is naturally an A -module and we can define the pullback functor

$$f^* : \text{QCoh}(\text{Spec } A) \rightarrow \text{QCoh}(\text{Spec } B)$$

by tensor $M \mapsto B \otimes_A M$. For a prestack Y , we define

$$\text{QCoh}(Y) = \lim_{S \xrightarrow{y} Y} \text{QCoh}(S).$$

In other words, an object $\mathcal{F} \in \text{QCoh}(Y)$ is a family of modules $\mathcal{F}_{S,y}$ with compatible isomorphisms

$$\mathcal{F}_{S',y'} \cong g^* \mathcal{F}_{S,y}.$$

Pullback between prestacks can be defined tautologically.

The pushforward functor is defined as the right adjoint of pullback. In some good cases, for example, if $f : X \rightarrow Y$ is a quasi-compact morphism between schemes, then f_* is continuous and satisfies the base change property.

Remark 1.6. A topological space K induces naturally a derived stack. Since K can be regarded as a ∞ -groupoid, we can sheafify the constant derived prestack

$$A \mapsto K$$

to a derived stack which we will still denote as K .

2 Hochschild-Kostant-Rosenberg theorem

We recall the statement of the Hochschild-Kostant-Rosenberg theorem.

Theorem 2.1. *Let A be a commutative finitely generated smooth k -algebra. Then*

$$\text{HH}_*(A) \cong \Lambda_A(\text{Der}(A)).$$

We will generalize this result to the case of derived schemes with affine diagonal. (I will assume this condition through the talk for convenience.)

We first recall the definition of Hochschild homology. Let \mathcal{C} be a complete, cocomplete, stable and representable dg category. We further assume that \mathcal{C} is dualizable which means there is another such category \mathcal{C}^\vee and functors $co : \text{Vect} \rightarrow \mathcal{C}^\vee \otimes \mathcal{C}$ and $ev : \mathcal{C} \otimes \mathcal{C}^\vee \rightarrow \text{Vect}$ satisfying axioms for evaluation and coevaluation.

Remark 2.2. We can talk about dualizability for objects in a monoidal category. In that case, $\mathcal{C}^\vee \otimes \mathcal{C} \not\cong \mathcal{C} \otimes \mathcal{C}^\vee$ in general.

For an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$. The trace of F is defined by the composition

$$\text{tr}(F) : \text{Vect} \xrightarrow{co} \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{\text{id}_{\mathcal{C}^\vee} \otimes F} \mathcal{C}^\vee \otimes \mathcal{C} \cong \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{ev} \text{Vect}$$

applying to $k \in \text{Vect}$. The Hochschild homology of \mathcal{C} is defined to be $\text{HH}(\mathcal{C}) := \text{tr}(\text{id}_{\mathcal{C}})$ the trace of the identify functor.

Now consider the case $\mathcal{C} = \text{QCoh}(X)$ for some smooth derived scheme X . The category $\text{QCoh}(X)$ is self-dual and $\text{QCoh}(X) \otimes \text{QCoh}(X) \cong \text{QCoh}(X \times X)$. Under these identification, the evaluation and coevaluation are given by the compositions

$$co : \text{Vect} \cong \text{QCoh}(*) \xrightarrow{p^*} \text{QCoh}(X) \xrightarrow{\Delta_*} \text{QCoh}(X \times X)$$

and

$$ev : \text{QCoh}(X \times X) \xrightarrow{\Delta^*} \text{QCoh}(X) \xrightarrow{p_*} \text{QCoh}(*) \cong \text{Vect},$$

where $p : X \rightarrow *$ is the projection to the point and $\Delta : X \rightarrow X \times X$ is the diagonal map. By definition, the Hochschild homology is $\text{HH}(\text{QCoh}(X)) = p_* \Delta^* \Delta_* p^*(k)$. (Note this statement holds true for a general stack.)

There is a way to re-write the expression on the right hand side. Recall that S^1 can be fit into the pushout diagram:

$$\begin{array}{ccc} * \amalg * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^1 \end{array}$$

Apply $\text{Map}(\cdot, X)$ to this diagram and we get the pullback diagram

$$\begin{array}{ccc} \text{Map}(S^1, X) & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

Here, for any two stack Y, Y' , we can form the mapping stack $\text{Map}(Y, Y')$ whose S points is the space

$$\text{Map}(Y, Y')(S) := \text{Map}_{\text{DSt}}(Y \times \text{Spec } S, Y').$$

People usually call the space $\text{Map}(S^1, X)$ the loop space of X and denote it as $\mathcal{L}X$. We can extend this diagram to the following one.

$$\begin{array}{ccccc}
\mathrm{Map}(S^1, X) & \xrightarrow{\pi} & X & \xrightarrow{p} & * \\
\pi \downarrow & & \Delta \downarrow & & \\
X & \xrightarrow{\Delta} & X \times X & & \\
p \downarrow & & & & \\
* & & & &
\end{array}$$

Here $\pi : \mathcal{L}X \rightarrow X$ is the canonical projection. We compute by base change that the Hochschild homology

$$\mathrm{HH}_*(\mathrm{QCoh}(X)) \cong p_*\Delta^*\Delta_*p^*(k) \cong p_*\pi_*\pi^*p^*(k) \cong p_*\pi_*\mathcal{O}_{\mathcal{L}X} \cong \mathcal{O}(\mathcal{L}X)$$

is isomorphic to the global functions of the loop space $\mathcal{L}X$. This is the first half of the Hochschild-Kostant-Rosenberg theorem and it holds for any stacks which satisfies the base change property. In the case of schemes, loop spaces and tangent bundles are isomorphic to each other.

3 The cotangent complex

As in the classical case, the cotangent complex plays a role in deformation theory.

Definition 3.1. A k -linear map $\delta : A \rightarrow M$ is a derivation if it satisfies graded Leibniz rule. We denote the k -module of all derivations from A to M by $\mathrm{Der}(A, M)$. The functor $M \mapsto \mathrm{Der}(A, M)$ is corepresentable by the Kähler differential Ω_A which can be constructed by generators and relations as in the classical case.

Remark 3.2. The Kähler differential Ω_A^1 doesn't respect quasi-isomorphism in general. But this construction does when the k -algebras are quasi-free and for any k -algebra A of finite type, there is a resolution $A' \rightarrow A$ which is quasi-free.

In the case $M \in A\text{-Mod}^{\leq 0}$, we denote the trivial square zero extension by $A \oplus M$ where the multiplication is given by $(a_1, m_1) \cdot (a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)$. One can check that the morphisms between cdgas $B \rightarrow A \oplus M$ over A correspond to morphisms between A -modules $f^*\Omega_B^1 \rightarrow M$.

More generally, given a prestack $X : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \rightarrow \mathcal{S}$ and a point $x : \mathrm{Spec} A = \mathcal{U} \rightarrow X$ and $F \in \mathrm{QCoh}(\mathcal{U})^{\leq 0}$, we can look at maps between $\mathcal{U}_F = \mathrm{Spec}(A \oplus F)$ and X below \mathcal{U} . This gives a covariant functor

$$F \mapsto \mathrm{Map}_{\mathrm{DSt}, \mathcal{U}/}(\mathcal{U}_F, X) = \mathrm{Map}_{\mathrm{DSt}}(\mathcal{U}_F, X) \times_{\mathrm{Map}_{\mathrm{DSt}}(\mathcal{U}, X)} \{x\}.$$

That is, the cotangent space at x fits into the fiber diagram.

$$\begin{array}{ccccc}
\mathrm{Hom}(T_x^*(X), F) & \xrightarrow{\sim} & \mathrm{Map}_{\mathrm{DSt}, \mathcal{U}/}(\mathcal{U}_F, X) & \longrightarrow & \mathrm{Map}_{\mathrm{DSt}}(\mathcal{U}_F, X) \\
& & \downarrow & & \downarrow \\
& & x & \longrightarrow & \mathrm{Map}_{\mathrm{DSt}}(\mathcal{U}, X)
\end{array}$$

Definition 3.3. We say X has a cotangent space at x if the above functor is corepresentable in $\mathrm{QCoh}(\mathcal{U})^-$ and we denote the corepresentative $T_x^*(X)$

If there is morphism between affine scheme $f : \mathcal{U} \rightarrow \mathcal{V}$ that sends a point $y : \mathcal{V} \rightarrow X$ to $x : \mathcal{U} \rightarrow X$, then there is a morphism between quasi-coherent sheaves

$$T_x^*(X) \rightarrow f^*T_y^*(X).$$

Definition 3.4. If X has all cotangent spaces and the above morphisms are all isomorphic. Then it gives a quasi-coherent sheaf

$$T^*(X) \in \mathrm{QCoh}(X)$$

and we call it the cotangent complex of X .

The cotangent complex doesn't always exist. For an affine scheme $X = \mathrm{Spec} A$, the Kähler differential Ω_A gives a construction for the cotangent complex. So it exists for Artin stacks. The way to construct it is by taking an smooth affine cover and use descent to get a global object.

Example 3.5. Let $F : \mathbb{A}^n \rightarrow \mathbb{A}^m$ be a polynomial map $F = (f_1, \dots, f_m)$ and denote $X = \mathrm{Spec} A = F^{-1}(0)$. Since fiber product is given by tensor product,

$$A = k[x_1, \dots, x_n] \otimes_{k[y_1, \dots, y_m]} k$$

where y_i acts on $k[x_1, \dots, x_n]$ by f_i . Denote $V = k\langle y_1, \dots, y_m \rangle$ and resolve k by the standard Koszul resolution

$$0 \rightarrow \Lambda^m V^* \otimes S(V) \rightarrow \Lambda^{m-1} V^* \otimes S(V) \rightarrow \dots \rightarrow V^* \otimes S(V) \rightarrow S(V) \rightarrow k \rightarrow 0$$

where the differential is given by

$$\omega \otimes f(y_1, \dots, y_m) \mapsto \sum_{i=0}^i (\iota_{y_i} \omega) \otimes y_i f(y_1, \dots, y_m).$$

This includes the case of intersecting the origin with itself. The resulting algebra A is the based loop space at $\{0\}$ which is usually denoted as $\mathbb{A}^m[1]$.

Then we see that $A \cong k[x_1, \dots, x_n, y_1, \dots, y_m]$ the quasi-free cdga with generators x_j 's in $\mathrm{deg} 0$, y_i 's in $\mathrm{deg} -1$ and the differential is given by $d_A y_j = f_j(x_1, \dots, x_n)$.

For this kind of algebra A , the cotangent complex $T^*(X)$ is given by the Kähler differential

$$\Omega_A = \langle d_{dR} x_1, \dots, d_{dR} x_n, d_{dR} y_1, \dots, d_{dR} y_m \rangle$$

with $\mathrm{deg}(d_{dR} x_i) = 0$, $\mathrm{deg}(d_{dR} y_j) = -1$, and the differential is given by

$$d_\Omega(d_{dR} y_i) = \sum_{j=0}^m \frac{\partial f_j}{\partial x_j} d_{dR} x_j.$$

One can also compute this by using the fact that taking cotangent complex is contravariant.

When F is smooth, the fiber is smooth and

$$\Omega_A \cong \Omega_{k[x_1, \dots, x_n]/(f_1, \dots, f_m)}[0]$$

where the later Ω denotes the classical Kähler differential.

Definition 3.6. The odd tangent bundle of a derived stack X is the linear derived stack

$$\mathbb{T}_X[-1] = \mathrm{Spec}_{\mathcal{O}_X} \mathrm{Sym}^\bullet(T^*(X)[1]).$$

Now we can state the theorem.

Theorem 3.7. *For X a derived scheme, the loop space $\mathcal{L}X$ is identified with the odd tangent bundle $\mathbb{T}_X[-1]$ such that constant loop correspond to the zero section. Equivalently, we have an equivalence of \mathcal{O}_X -algebras*

$$\mathcal{O}_{\mathcal{L}X} \cong \Omega_X^-.$$

Take global section on both side and use the fact that $\mathrm{HH}_*(\mathrm{QCoh}(X)) \cong \mathcal{O}(\mathcal{L}X)$, we get

$$\mathrm{HH}_*(\mathrm{QCoh}(X)) \cong \Gamma(X; \Omega_X^-)$$

We need to introduce one more definition before proving the theorem. For a (not necessarily coconnective) commutative dg k -algebras R , we can assign an affine stack $\mathrm{Spec}(R)$ to it whose S points is given by the space

$$\mathrm{Spec}(R)(S) := \mathrm{Hom}_{DGA}(R, S).$$

For a stack X , the affinization of X is the universal affine stack $\mathrm{Aff}(X)$ such that

$$\mathrm{Map}(X, \mathrm{Spec} S) \cong \mathrm{Map}(\mathrm{Aff}(X), \mathrm{Spec} S)$$

for all algebras S . Now consider the case K which is a nice topological space, for example, a CW complex. In the category of groupoids, we can always write $K = (\mathrm{colim}_K *)$ a colimit of points by picking a triangulation. Then we compute

$$\mathcal{O}(K) = \mathcal{O}(\mathrm{colim}_K *) = \lim_K \mathcal{O}(*) = \lim_K k = C^*(K, k).$$

The key point is that taking global section from the category of stack sends colimits to limits and the k 's glues back to $C^*(K, k)$. So the global functions of S^1 is

$$\mathcal{O}(S^1) = C^*(S^1, k) \cong H^*(S^1, k) = k[\eta]/(\eta^2)$$

with $|\eta| = 1$. Here we use the formality of $C(S^1, k)$ meaning this dg algebra is isomorphic to its cohomology algebra. Note k is of characteristic 0 is needed for this fact to be true. Then we will have the following lemma.

Lemma 3.8. *The affinization of S^1 is $\mathrm{Aff}(S^1) \cong \mathrm{Spec} k[\eta]$ where $\deg \eta = 1$. We denote the latter $\mathbb{A}^1[1]$ and call it the odd affine line.*

Proof. The following two functor forms an adjunction pair,

$$\mathcal{O} : DSt_k \rightleftarrows DGA_k^{op} : \mathrm{Spec}.$$

□

The second lemma we need is that in the case of schemes, taking loop space satisfies Zariski codescent.

Lemma 3.9. *For a scheme X , and $U \hookrightarrow X$ a Zariski open subscheme, the induced map $\mathcal{L}U \hookrightarrow \mathcal{L}X$ is also Zariski open. The assignment of loops*

$$U \mapsto \mathcal{L}U$$

forms a cosheaf on the Zariski site of X .

Proof. Affineness allows us to conclude $\mathcal{L}U \cong \mathcal{L}X \times_X U$ by computing $u^* \mathcal{O}_{\mathcal{L}X} \cong \mathcal{O}_{\mathcal{L}U}$ where $u : U \hookrightarrow X$ is the inclusion. □

4 Proof of the theorem

Proof. We can assume $X = \text{Spec } R$ is affine by the second lemma and in this case $\mathcal{L}X \cong \text{Map}_{\text{DSt}}(\mathbb{A}^1[-1], X)$ by the first lemma. So our goal is to show $\text{Map}(\mathbb{A}^1[-1], X) \cong \mathbb{T}_X[-1]$ in a natural way.

Take any $S \in \text{DGA}^-$ and compute

$$\text{Map}(\mathbb{A}^1[-1], X)(S) = \text{Map}_{\text{DSt}}(\text{Spec } S \times \mathbb{A}^1[-1], X) = \text{Hom}_{\text{DGA}}(R, S \oplus S[-1]).$$

Recall that for any S -point $x : \text{Spec } S \rightarrow X$ and $M \in S - \text{Mod}^-$, there is a fiber diagram.

$$\begin{array}{ccc} \text{Hom}_S(T_x^*(X), M) & \longrightarrow & \text{Hom}_{\text{DGA}^-}(R, S \oplus M) \\ \downarrow & & \downarrow \\ x & \longrightarrow & \text{Hom}_{\text{DGA}^-}(R, S) \end{array}$$

Some connectivity estimate shows us that the following diagram is still a fiber diagram.

$$\begin{array}{ccc} \text{Hom}_S(T_x^*(X), S[-1]) & \longrightarrow & \text{Hom}_{\text{DGA}^-}(R, S \oplus S[-1]) \\ \downarrow & & \downarrow \\ x & \longrightarrow & \text{Hom}_{\text{DGA}^-}(R, S) \end{array}$$

Then, we can shift the grading and use the adjunction for symmetric algebras to get

$$\text{Hom}_S(T_x^*(X), S[-1]) \cong \text{Hom}_S(T_x^*(X)[1], S) \cong \text{Hom}_{\text{DGA}_S^-}(\text{Sym}^*(T_x^*(X)[1]), S).$$

This identification is natural in the sense that if S' and $y : \text{Spec } S' \rightarrow X$ is another pair of such data and there is a morphism $f : \text{Spec } S \rightarrow \text{Spec } S'$ such that $x = y \circ f$, then f pulls back the whole diagram functorially. This implies the isomorphisms glue to a global isomorphism $\text{Map}(\mathbb{A}^1[-1], X) \cong \mathbb{T}_X[-1]$. \square

Remark 4.1. The loop space $\mathcal{L}X \rightarrow X$ is a group object over X and taking the Lie algebra gives the odd tangent complex $\mathbb{T}_X[-1] \rightarrow X$. In the case of schemes, we can think of the isomorphism as the inverse exponential map \exp^{-1} . If X is not a scheme, these two spaces might not be isomorphic to each other. For example, when X is the classifying stack BG for some group G . The loop space $\mathcal{L}X$ is given by G/G the adjoint quotient stack but $\mathbb{T}_X[-1] = \mathfrak{g}/G$. See the original paper [BD10] for more discussions on loops, unipotent loops, and formal loops.

References

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