Recollections on Analytic Stacks

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October 30, 2025

Abstract

These notes are for the second seminar talk for ARGOS in Winter 2025. The goal of the seminar is to study the recent paper *Analytic de Rham stacks of Fargues-Fontaine curves* by J. Anschütz, G. Bosco, A-C. Le Bras, J. E. Rodríguez Camargo, and P. Scholze [2]. However, this talk is a general talk aiming to give a quick recollection of the theory of analytic stacks. Here, we mostly follow the notations and discussions from [7], which derives much of its content from the course [3]. The author apologizes if any errors that occur in the notes.

1 Introduction and condensed mathematics

The purpose of these notes is to give a quick recollection of the theory of analytic stacks. Roughly speaking, there is a notion of analytic rings, which consists of condensed rings, a version of topological rings (but with better homological algebraic properties), equipped with a notion of completeness. They are the affine objects in this analytic world. As one will see in these notes, the theory of analytic rings has an inbuilt notion of six-functor formalism; the most natural topology is thus the so-called D-topology, a universal Grothendieck topology that allows very strong descent properties. Roughly, an analytic stack should be a hypersheaf in anima with regard to this topology. However, at the moment one has to consider a slight enlargement that sits between sheaves and hypersheaves. One main reason is that the theory of analytic stacks is supposed to contain more classical objects such as algebraic stacks, Betti stacks, and adic spaces.[1]

Now, we begin the discussion with condensed sets. The main purpose of the condensed formalism is to have a framework where topology and homological algebra coexist harmoniously.

Example 1.1. The category of topological abelian groups is not abelian. For example, the inclusion $\mathbb{R}_{disc} \hookrightarrow \mathbb{R}$ has a zero cokernel but is not an isomorphism.

To fix this issue, one introduces a larger category that contains the theory of (good) topological spaces, which we will refer to as the condensed sets, CondSet, and it will be defined as the sheaves over a certain site. Denote by Pro(Fin) the category of profinite sets,

[1] And this ad hoc construction is needed for the Betti stacks.

defined as the pro-completion of the category of finite sets. Its objects are formal inverse limits "$\lim_{i\in I^{op}} S_i$" and the Hom-set is given by

$$\mathrm{Hom}_{\mathrm{Pro(Fin)}}(\text{“}\lim_{i\in I^{op}} S_i\text{”}, \text{“}\lim_{j\in J^{op}} T_j\text{”}) = \lim_{j\in J^{op}} \operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathrm{Fin}}(S_i, T_j)$$

Now, for such an inverse system of finite sets "$\lim_{i\in I^{op}} S_i$", one can put a limit topology on $S := \lim_{i\in I^{op}} S_i$ by viewing the S_i's as discrete sets.

Lemma 1.2. *The assignment* Pro(Fin) $\to$ Top *by* "$\lim_{i\in I^{op}} S_i$" $\mapsto \lim_{i\in I^{op}} S_i$ *is fully-faithful and the essential image consists of totally disconnected compact Hausdorff spaces.*

Recall that a topological space X is called totally disconnected if all connected components are of the form $\{x\}$ for some $x \in X$. We will use Prof to denote the category of totally disconnected compact Hausdorff spaces with continuous maps, and freely interchange the two perspectives when mentioning a profinite set. For the purpose of analytic geometry, we will actually use a smaller category, the light profinite set $\mathrm{Prof}^{\mathrm{light}}$.

Lemma 1.3. *The topological space* $\lim_{i\in I^{op}} S_i \in$ Prof *is second-countable if and only if* "$\lim_{i\in I^{op}} S_i$" *is isomorphic a sequential limit of finite sets. In this case, we call it a light profinite set.*

Example 1.4. The one point compactification of the integer $\mathbb{N} \cup \{\infty\}$ is a light profinite set since it can be written as the limit $\mathbb{N} \cup \{\infty\} = \lim_{n\in\mathbb{N}^{op}} \{0, \cdots, n, \infty\}$ where the transition map from $\{0, \cdots, n+1, \infty\}$ to $\{0, \cdots, n, \infty\}$ is given by the identity on every thing except $n+1$ and $n+1$ is sent to ∞.

Since we will be considering only the light setting, we will drop the adjective and superscript "light" from now on.

Definition 1.5. The condensed sets CondSet is the category of set-valued sheaves on Prof with respect to the Grothendieck topology generated by finitely jointly surjective covers. In other words, the basic cover is of the form $\{f_i : T_i \to S\}$ such that $\amalg f_i : \amalg T_i \to S$ is surjective. More precisely, a presheaf X : Prof $\to$ Set is a sheaf if

1. $X(\varnothing) = \{*\}$.
2. $X(S_1 \amalg S_2) = T(S_1) \times T(S_2)$
3. For any surjection $S_1 \to S_2$, we have

$$T(S_2) = eq\left(T(S_1) \rightrightarrows T(S_1 \times_{S_2} S_1)\right).$$

Example 1.6. There is a condensification functor Top $\to$ Cond(Set) sending a topological space X to the condensed set $\underline{X}(S) := \mathrm{Cont}(S, X)$. For a Hausdorff space X, $\underline{X}(\mathbb{N} \cup \{\infty\})$ is equivalent to the set of convergent sequences in X, and the restriction map $\underline{X}(\mathbb{N} \cup \{\infty\}) \to \underline{X}(\mathbb{N})$ is one-to-one.

More generally, one can define condensed abelian groups, condensed rings, etc. as sheaves valued in the corresponding category. Similarly, there are higher categorical analogues, but in this case one has to consider hypersheaves [7, Definition 4.0.1] instead of sheaves. Thus, the category of condensed anima $\mathrm{CondAni} := \mathrm{HypSh}(\mathrm{Prof}, \mathrm{Ani})$.

Notation 1.7. For the most part of the talk, we will work in the higher categorical setting, so following the convention in [7, Example 4.0.2], we will, for example, use CondRing to denote the category of condensed (animated) rings, and emphasize that a ring is static when it's relevant.

2 Analytic rings

The category of condensed (animated) rings CondRing can be thought of as a version of topological rings (with an extra homotopy direction). To discuss analytic structures, however, we will need to specify the notion of "completeness," and this can be done by singling out a subcategory of modules.

Definition 2.1 ([7, Definition 4.1.1])**.** An uncompleted analytic ring $A \in \mathrm{AnRing}^{uc}$ is a pair $(A^{\triangleright}, D(A))$ where $A^{\triangleright} \in \mathrm{CondRing}$ is a condensed ring and $D(A) \subseteq D(A^{\triangleright})$ is a subcategory of its derived category such that

1. $D(A) \subseteq D(A^{\triangleright})$ preserves limits and colimits, which in particular implies that $D(A)$ is presentable.

2. $D(A)$ is linear over $D(\mathrm{Cond\,Ab})$. More precisely, for all $C \in D(\mathrm{Cond\,Ab})$ and $M \in D(A)$, the object $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}}(C, M)$ is in $D(A)$.

3. The left adjoint $A \otimes_{A^{\triangleright}} (-)$ of $D(A) \subseteq D(A^{\triangleright})$ fixes the connective part of the standard t-structure $D_{\geq}(A^{\triangleright})$ (and thus there is an induced t-structure on $D(A)$).

4. We say A is complete if $A^{\triangleright} \in D(A)$. In this case, we simply say $A = (A^{\triangleright}, D(A))$ is an analytic ring and use the notation AnRing to denote this subcategory.

A morphism between (uncompleted) analytic rings $g : A \to B$ is a morphism $g^{\triangleright} : A^{\triangleright} \to B^{\triangleright}$ such that the restriction of the forgetful map $D(B^{\triangleright}) \to D(A^{\triangleright})$ to $D(B)$ factorizes through $D(A)$.

Proposition 2.2 ([7, Theorem 4.1.13], [5, Proposition 2.3.12])**.** *The inclusion* $\mathrm{AnRing} \hookrightarrow \mathrm{AnRing}^{uc}$ *has a left adjoint* $(-)^{=}$*.*

This proposition implies that AnRing admits small colimits: For a diagram $\{A_i\}_{i \in I}$ in AnRing, the colimit of the uncompleted analytic rings $(\operatorname{colim}_{i\in I} A_i^{\triangleright})^{uc}$ has the underlying ring given by $\operatorname{colim}_{i\in I} A_i^{\triangleright}$, and $M \in D(\operatorname{colim}_{i\in I} A_i^{\triangleright})$ is complete if its restriction to $D(A_i^{\triangleright})$ lies in $D(A_i)$ for all i. Then, the colimit analytic ring $\operatorname{colim}_{i\in I} A_i^{\triangleright} = \left((\operatorname{colim}_{i\in I} A_i^{\triangleright})^{uc}\right)^{=}$ is given by its completion.

Proposition 2.3 ([7, Proposition 4.1.14])**.** *The functor* D : AnRing $\to$ CAlg($\mathrm{Pr}^L_{D^{\mathrm{cond}}(\mathbb{Z})}$) *by* $A \mapsto D(A)$ *commutes with colimits. In particular, for* A, B*, and* $C \in$ AnRing*, we have* $D(B \otimes_A C) = D(B) \otimes_{D(A)} D(C)$*.*

This last equation is an instance of the *categorical Künneth formula* and will play a role in the six-functor formalism.

Example 2.4 ([7, Section 3])**.** The main example in non-archimedean geometry is the solid analytic structure: Note that the profinite set $\mathbb{N} \cup \{\infty\}$ has a monoid structure, as "+" extends by $n + \infty = \infty$. Thus, we can consider the group ring $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$. Set $P :=$ $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\{\infty\}$ to be the condensed abelian group of null sequences and write $\mathbb{Z}[\hat{q}]$ when viewing it as a ring. Here $\hat{q} = [1]$ corresponds to the shift [7, Remark 3.2.3]. The main feature of non-archimedean analysis (when complete) is that all null sequences $\{a_n\}$, sequences having the property $a_n \to 0$, have a convergent series $\sum a_n$. From this point of view, one can check that a topological abelian group M is complete if and only if

$$1 - q : \mathrm{Hom}(\mathbb{Z}[\hat{q}], \underline{M}) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}[\hat{q}], \underline{M}).$$

Indeed, the map $1 - q$ sends a null sequence $(a_0, a_1, a_2, \cdots)$ to $(a_0 - a_1, a_1 - a_2, \cdots)$, which is always one-to-one, and an inverse, if it exists, would have the form

$$(b_0, b_1, b_2, \cdots) \mapsto (\sum_{n \geq 0} b_n, \sum_{n \geq 1} b_n, \cdots).$$

Now, view $\mathbb{Z}$ as a condensed ring by $\underline{Z}$. Then one can show that, if we set $D(\mathrm{Cond}) \subseteq$ $D^{\mathrm{cond}}(\mathbb{Z})$ to be the subcategory consisting of M such that

$$1 - q : \mathrm{RHom}(\mathbb{Z}[\hat{q}], M) \xrightarrow{\sim} \mathrm{RHom}(\mathbb{Z}[\hat{q}], M),$$

$\mathbb{Z}$ is in fact in $D(\mathrm{Solid})$. Thus, we set $\mathbb{Z}_{\square} = (\mathbb{Z}, D(\mathrm{Solid})) \in$ AnRing. Since $\mathbb{Z}$ is solid and $D(\mathrm{Solid})$ has all limits and colimits, we in particular have that all discrete abelian groups are solid. But this further implies, for example, that the power series ring $\mathbb{Z}[[q]]$ and the p-adic numbers $\mathbb{Z}_p$, with their condensed structure given by

$$\mathbb{Z}[[q]] = \lim_{n \in \mathbb{N}^{op}} \mathbb{Z}[q]/(q^n) \text{ and } \mathbb{Z}_p := \lim_{n \in \mathbb{N}^{op}} \mathbb{Z}/p^n\mathbb{Z},$$

are solid [7, Section 3.5].

Remark 2.5. The main reason for using the light setting is that P as mentioned above will be an internal projective generator, which is not true when including profinite sets with larger cardinalities.

3 Six-functor formalism and the D-topoloy

The opposite category $\mathrm{AnStk}^{\mathrm{aff}} := \mathrm{AnRing}^{op}$, which we will refer as the category of analytic affinoid, will serve as the building blocks of analytic geometry. For $A \in$ AnRing, we use the symbol AnSpec$A$ to the denote the corresponding object in $\mathrm{AnStk}^{\mathrm{aff}}$. Since A comes with a presentable category $D(A)$, we would like to define a Grothendieck topology that is as refine as possible which makes this assignment a sheaf. This can in fact be done in a very general setting of six-functor formalism.

Example 3.1. Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. In addition to the usual star-adjunction $f^* \dashv f_*$, there is a shriek pushforward $f_! : \mathrm{Sh}(X; D(\mathbb{Z})) \to \mathrm{Sh}(Y; D(\mathbb{Z}))$, which in the case when $Y = \{*\}$ is given by compactly supported sections $\Gamma_c(X; -)$. For example, when X is a manifold, then $\Gamma_c(X; X_{\mathbb{R}}) = \mathrm{H}^*_{dR,c}(X)$ is given by the compactly supported de Rham cohomology. This functor satisfies the projection formula $F \otimes (f_! G) = f_!(f^* F \otimes G)$ and base change: Given a pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \big\downarrow{\scriptstyle g'} & \lrcorner & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array},$$

there is an equivalence $g^* f_! = f'_! g'^*$. For example, when $X = \{y\}$ is a point in Y, it means that the stalk $(g_! F)_y = \Gamma_c(g^{-1}(y); F)$ can be computed as the compactly supported cohomology of the fiber.

Definition 3.2 ([7, 6.1.1])**.** A geometric set-up is a pair $(\mathcal{C}, E)$ where $\mathcal{C}$ is an ∞-category with finite limits and E is a class of morphisms, which are "!-able," such that

(1) E contains isomorphisms.

(2) E is stable under composition and pullback.

(3) E is stable under taking diagonals.

For example, being stable under pullback means that if $f : X \to Y$ is in E and $Y' \to Y$ is any map in $\mathcal{C}$, then $f' : X' \to Y'$ as in Example 3.1 above is also in E.

Definition 3.3 ([7, 6.1.2])**.** Given a geometric set-up $(\mathcal{C}, E)$, the category of correspondences $\mathrm{Corr}(\mathcal{C}, E)^{\otimes}$ is a symmetric monoidal category, which contains the same objects as $\mathcal{C}$, while morphisms from X to Y are given by spans $X \xleftarrow{g} Z \xrightarrow{f} Y$ such that $f \in E$. Composition of two morphisms is given by the following diagram

$$\begin{array}{cc} U & \longrightarrow Z \\ \big\downarrow & \\ Y & \end{array} \;\circ\; \begin{array}{cc} W & \longrightarrow Y \\ \big\downarrow & \\ X & \end{array} \;=\; \begin{array}{ccccc} W \times_Y U & \longrightarrow & U & \longrightarrow & Z \\ \big\downarrow & \lrcorner & \big\downarrow & & \\ W & \longrightarrow & Y & & \\ \big\downarrow & & & & \\ X & & & & \end{array}$$

The symmetric monoidal structure is given by $X \times Y$, the underlying Cartesian product.

Definition 3.4. A three-functor formalism D on $(\mathcal{C}, E)$ is a lax symmetric monoidal functor $D : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}_{\infty}^{\times}$.

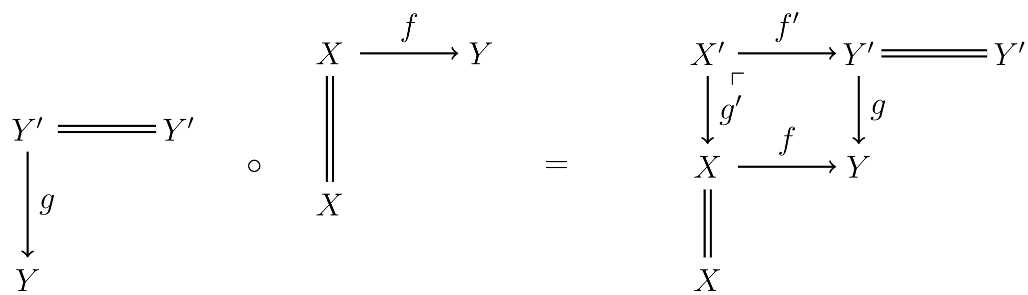
The category of correspondences $\mathrm{Corr}(\mathcal{C}, E)^{\otimes}$ is the universal category that captures base change and projection formula. For example, one notice that there is a functor

$$\mathcal{C}^{op} \to \mathrm{Corr}(\mathcal{C}, E)$$
$$Y \xleftarrow{g} Z \mapsto (Y \xleftarrow{g} Z = Z),$$

and we set $g^* := D(Y \xleftarrow{g} Z = Z)$. Denote by $\mathcal{C}_E$ the non-full subcategory consisting of only morphisms in E. We likewise have

$$\mathcal{C}_E \to \mathrm{Corr}(\mathcal{C}, E)$$
$$X \xrightarrow{f} Y \mapsto (X = X \xrightarrow{f} Y),$$

and we set $f_! = D(X = X \xrightarrow{f} Y)$. Apply D to the following composition



and we get $g^* f_! = f'_! g'^*$. One can simiarly obtain the $\otimes$ and the projection formula similarly by using the symmetric monoidal structure.

Definition 3.5. A six-functor formalism is a three-functor formalisms where f^*, $f_!$ and $\otimes$ admits right adjoints f_*, $f^!$ and $\underline{\mathrm{Hom}}(-,-)$.

Now, we can define the D-topology.

Definition 3.6 ([7, 6.1.4])**.** Let $(\mathcal{C}, E)$ be a geometric set-up and D a six-functor formalism.

(a) A map $f : X \to Y$ in $\mathcal{C}$ is said to satisfy D^*-descent if

$$D^*(Y) \xrightarrow{\sim} \lim \left(\prod_{i \in I} D^*(X) \rightrightarrows D^*(X \times_Y X) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

can be computed by the totalization of the Čech nerve. Here, we use D^* to indicate that the functors are given by upper-$*$'s. We say f satisfies universal D^*-descent if the equivalence holds under all pullbacks $Y' \to Y$ in $\mathcal{C}$.

(b) A map $f : X \to Y$ in E is said to satisfy $D^!$-descent if

$$D^!(Y) \xrightarrow{\sim} \lim \left(\prod_{i \in I} D^!(X) \rightrightarrows D^!(X \times_Y X) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

can be computed by the totalization of the Čech nerve. Here, we use $D^!$ to indicate that the functors are given by upper-!'s. We say f satisfies universal $D^!$-descent if the equivalence holds under all pullbacks $Y' \to Y$ in $\mathcal{C}$.

(c) ([7, Example 6.2.10]). The D-topology on $\mathcal{C}$ is generated by sub-canonical covers $\{f_i : Y_i \to X_i\}_{i\in I}$, $|I| < \infty$, such that $\amalg f_i : \amalg Y_i \to X$ satisfies universal D^* and $D^!$-descent. Recall that such a cover is sub-canonical if for any $Z \in \mathcal{C}$, the map

$$\operatorname{Hom}(X, Z) \xrightarrow{\sim} \lim \left(\prod_{i\in I} \operatorname{Hom}(\amalg Y_i, Z) \rightrightarrows \operatorname{Hom}\left((\amalg Y_i) \times_X (\amalg Y_i), Z\right)\right) \substack{\to\\\to\\\to} \cdots \right)$$

is an equivalence.

Proposition 3.7 ([7, Proposition 6.3.7], [9, Theorem 5.12])**.** *Let $(\mathcal{C}, E)$ be a geometric set-up, and $D : \operatorname{Corr}(\mathcal{C}, E) \to \operatorname{Cat}_\infty$ be a six-functor formalism such that*

(1) D is presentable. That is, D factorizes to $D : \operatorname{Corr}(\mathcal{C}, E) \to \mathrm{Pr}^{\mathrm{L}}$, the category of presentable categories with morphisms being left adjoints.

(2) D satisfies the categorical Künneth formula. That is, $D(X \times Y) = D(X) \otimes D(Y)$.

Then, a finite family $\{Y_i \to X\}$ satisfies universal D^* and $D^!$-descent if and only if it satisfies $D^!$-descent.

Remark 3.8. In fact, [9, Theorem 5.12] does not require the categorical Künneth assumption, though it has a weaker conclusion. One can use it to deduce the version here.

Definition 3.9. In the same setting as Definition 3.6, the Grothendieck topology we obtain by replacing the requirement of universal D^* and $D^!$-descent with only $D^!$-descent, or simply !-descent when the context is clear, is called the !-topology.

Now, our goal is to explain that there is a six-functor formalism for analytic affinoids $\mathrm{AnStk}^{\mathrm{aff}}$ such that the D-topology coincides with the !-topology. To motivate the class E, we note that while we have explained what a six-functor formalism is, we have not touched upon its existence. In practice, a convenient situation is when E admits a certain *suitable decomposition* I and P, which play the role of "open immersion" and "proper map". We omit what assumptions are needed for I and P to be suitable, except to mention that any map f in E should admit a decomposition $f = \bar{f} \circ j$ for some $j \in I$ and $\bar{f} \in P$. In addition, as motivated by the next example, we should have $j_! \dashv j^*$ and $p_! \dashv p^*$. See [4, Proposition 1.2.5] for details.

Example 3.10. In the situation of Example 3.1, constructing the $*$-functoriality is relatively straightforward. Let $f : X \to Y$. We've seen that $f_!$ is supposed to be taking fiberwise compactly supported cohomology, thus we should have $f_! = f_*$ when f is proper. Similarly, if there is a decomposition of $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ by an open set U and its complement, it is not hard to see that there exists a recollement,

$$\operatorname{Sh}(Z) \overset{i_*}{\hookrightarrow} \operatorname{Sh}(X) \overset{j^*}{\twoheadrightarrow} \operatorname{Sh}(U),$$

meaning that both i_* and j^* have both left and right adjoints. Set $j_! \dashv j^*$. Now, for a general $f : X \to Y$, f factorizes to $X \xrightarrow{j} \overline{X} \xrightarrow{\bar{f}} Y$ by using, for example, Stone-Čech compactification, and $f_!$ can be computed as $f_! = \bar{f}_* j_!$.

For $\mathrm{AnStk}^{\mathrm{aff}}$, we define P, I, and E as follows. Let $f : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$ be a morphism and set $D(B_{A/}) := D(B^{\triangleright}) \times_{D(A^{\triangleright})} D(B)$. The factorization assumption for $f_* : D(B^{\triangleright}) \to D(A^{\triangleright})$ implies that there is a commutative diagram:

$$\begin{array}{ccccc} D(B) & \hookrightarrow & D(B_{A/}) & \hookrightarrow & D(B^{\triangleright}) \\ & \searrow & \downarrow & & \downarrow \\ & & D(A) & \hookrightarrow & D(A^{\triangleright}) \end{array}$$

Note that $(B^{\triangleright}, D(B_{A/}))$ defines an analytic structure on $B^{\triangleright}$, which we will write as $B_{A/}$ and call the induced analytic structure.

Definition 3.11 ([7, Definition 6.3.2])**.** Following the notation of the above diagram, we have:

(a) $f \in P$ if $D(B) = D(B_{A/})$.

(b) $f \in I$ if $f^* : D(A) \to D(B)$ has furthermore a fully faithful left adjoint $f_!$. (This would imply f_* is fully faithful as well.)

(c) $f \in E$ if the factorization $\mathrm{AnSpec}B \xrightarrow{j} \mathrm{AnSpec}B_{A/} \to \mathrm{AnSpec}A$ is such that $j \in I$.

Corollary 3.12. *The assignment*

$$\begin{aligned} D : \mathrm{AnStk}^{\mathrm{aff},op} &\to \mathrm{Pr}^{\mathrm{L}} \\ \mathrm{AnSpec}A &\mapsto D(A) \end{aligned}$$

extends to a presentable six-functor formalism $D : \mathrm{Corr}(\mathrm{AnStk}^{\mathrm{aff}}, E)^{\otimes} \to \mathrm{Pr}^{\mathrm{L},\otimes}$ *which satisfies the categorical Künneth formula. In particular, the D-topology coincides with the !-topology in this case.*

4 Analytic stacks

The topos of analytic stacks is somewhat subtle, as it will be sitting between sheaves and hypersheaves. Similar to the case of defining the D-topology, it can be done in a fairly general setting: Let $(\mathcal{C}, E)$ be a geometric set-up and set $\widetilde{\mathcal{C}} := \mathrm{Sh}(\mathcal{C})$ to be the topos of sheaves with respect to the D-topology, where $D : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Pr}^{\mathrm{L}}$ is a presentable six-functor formalism. One can extend the geometric set-up $(\mathcal{C}, E)$ along the Yoneda inclusion $\mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$: First define $\widetilde{E}_0$ by declaring $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ in $\widetilde{\mathcal{C}}$ to be in $\widetilde{E}_0$ if for any map $Y \to \widetilde{Y}$ such that $Y \in \mathcal{C}$, we have $X := \widetilde{X} \times_{\widetilde{Y}} Y$ to be in $\mathcal{C}$ and the induced map $f : X \to Y$ to be in E.

Theorem 4.1 ([7, Theorem 6.2.11], [4, Theorem 3.4.11])**.** *There exists a minimal class* $\widetilde{E} \supseteq \widetilde{E}_0$ *such that* $\widetilde{E}$ *is stable under disjoint union, local on the target and source, and tame. Furthermore,* D *extends to a six-functor formalism*

$$\mathrm{Corr}(\widetilde{\mathcal{C}}, \widetilde{E})^{\otimes} \to \mathrm{Pr}^{\mathrm{L}}.$$

In this case, the category $D(\widetilde{X})$ for $X \in \widetilde{\mathcal{C}}$ can be computed by the formula

$$D(\widetilde{X}) = \lim_{\substack{X \in \mathcal{C} \\ X \to \widetilde{X}}} D(X).$$

Remark 4.2. Heyer and Mann's theorem in fact holds more generally for more general Grothendieck topologies as long as D is sheafy. That is, $D^* : \mathcal{C}^{op} \to \mathrm{Cat}_\infty$ satisfies descent. The D-topology is just the universal case.

Lemma 4.3 ([7, Lemma 6.3.22])**.** *If there is a map $(\mathcal{D}, F) \to (\mathcal{C}, E)$ between geometric set-ups, then there is an induced functor $\mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ between sheaves with respect to the D-topology. Furthermore, it is left exact and colimit-preserving.*

Now, for any geometric set-up $(\mathcal{C}, E)$ we will define a category $\widehat{\mathrm{Sh}}(\mathcal{C})^2 \subseteq \widetilde{\mathcal{C}}$ that has a stronger descent property, which we will discuss later. For this purpose, we will assume the six-functor formalism D on $(\mathcal{C}, E)$ satisfies the following Hypothesis 4.4. First, for any diagram category I, we write $I^{\triangleright} = I \cup \{\varnothing\}$ for the diagram category with an added initial point $\varnothing$.

Hypothesis 4.4 ([7, Hypothesis 6.3.10])**.** For any given $I \to \mathcal{C}^{op}$, we assume that the condition of

$$D(X_\varnothing) \to \lim_{i \in I^{op}} D^*(X_i)$$

being an isomorphism implies the condition of

$$\mathrm{Hom}(X_\varnothing, Y) \to \lim_{i \in I^{op}} \mathrm{Hom}(X_i, Y)$$

being an isomorphism for all $Y \in \mathcal{C}$. In particular, $\{f_i : Y_i \to X\}$ is a sub-canonical cover if it satisfies $*$-descent.

Lemma 4.5 ([7, Lemma 6.3.23])**.** *The geometric set-up $(\mathrm{AnStk}^{\mathrm{aff}}, E)$ satisfies Hypothesis 4.4.*

Definition 4.6. We say a map $f : X \to Y$ in $\mathrm{Sh}(\mathcal{C}_E)$ is a !-equivalence if any pullback of any iterated diagonal of f is sent to an isomorphism under $D^!$. That is, $f^! : D(Y) \xrightarrow{\sim} D(X)$, $\Delta_f^! : D(X \times_Y X) \xrightarrow{\sim} D(X)$, $D(X \times_{X \times_Y X} X) \xrightarrow{\sim} D(X)$, etc.

Now, as Lemma 4.3 is applicable to the tautological map $(\mathcal{C}_E, E) \to (\mathcal{C}, E)$, we obtain a functor $\mathrm{Sh}(\mathcal{C}_E) \to \mathrm{Sh}(\mathcal{C})$.

Definition 4.7 ([7, Definition 6.3.14])**.**

(a) A map $f : X \to Y$ is a !-equivalence if for any iterated diagonal $X \to X^{(n)_Y}$ and any $Z \in \mathcal{C}$, the induced map $X \times_{X^{(n)_Y}} Z \to Z$ is locally in the !-topology an arrow that comes from a !-equivalence in $\mathrm{Sh}(\mathcal{C}_E)$ under the functor $\mathrm{Sh}(\mathcal{C}_E) \to \mathrm{Sh}(\mathcal{C})$.

[2] In [7], the category defined here is denoted as $\widetilde{\mathrm{Sh}}(\mathcal{C})$. However, Scholze suggested the notation $\widehat{\mathrm{Sh}}(\mathcal{C})$ during the seminar.

(b) We define $\widehat{\mathrm{Sh}}(\mathcal{C}) \subseteq \mathrm{Sh}(\mathcal{C})$ to be the subcategory localized at ∞-connective !-equivalences.

Note that by inverting along !-equivalences which are ∞-connective, we automatically have $\mathrm{HypSh}(\mathcal{C}) \subseteq \widehat{\mathrm{Sh}}(\mathcal{C})$. As D extends to the entire $\mathrm{Sh}(\mathcal{C})$, we can restrict it to $\widehat{\mathrm{Sh}}(\mathcal{C})$. Tautologically, $D^!$ defines a sheaf on it. But we in fact have the same property for D^* as well.

Lemma 4.8 ([7, Proposition 6.3.17])**.** *D^* localizes along ∞-connective !-equivalences. Thus, D^* descends to a sheaf,*

$$D^* : \widehat{\mathrm{Sh}}(\mathcal{C}) \to \mathrm{Pr}^{\mathrm{L}}.$$

Definition 4.9. We define AnStk by picking $\mathcal{C} = \mathrm{AnStk}^{\mathrm{aff}}$ with E as in Definition 3.11. [3]

Lemma 4.10 ([7, Lemma 6.3.22])**.** *Let $(\mathcal{D}, F) \to (\mathcal{C}, E)$ be a map between geometric set-ups and D a presentable six-functor formalism on $(\mathcal{C}, E)$ such that it and its restriction to $(\mathcal{D}, F)$ satisfy Hypothesis 4.4. Then, there is a functor*

$$\widehat{\mathrm{Sh}}(\mathcal{D}) \to \widehat{\mathrm{Sh}}(\mathcal{C})$$

which is left exact and colimit-preserving.

5 Examples

The theory of analytic stacks contains many well-known frameworks such as algebraic stacks, locally compact Hausdorff spaces, and adic spaces. We first recall that algebraic stacks are built out of affine schemes $\mathrm{Sch}^{\mathrm{aff}} := (\mathrm{Ring}^{op})$ with E given by all maps. There are two canonical ways to view a ring A as an analytic ring. The vanilla way is to simply recall that static rings embed into condensed rings by viewing them as discrete topological spaces:

$$\begin{aligned} (-)^{\mathrm{cond}} : \mathrm{Ring} &\to \mathrm{CondRing} \\ A &\mapsto \underline{A}(S) := \mathrm{Cont}(S, A) = \operatorname*{colim}_{n\in\mathbb{N}} A^{S_n}. \end{aligned}$$

Similarly, we can consider the condensed A-modules simply as $D^{\mathrm{cond}}(A) = D(A) \otimes_{D(\mathbb{Z})} D(\mathbb{Z}^{\mathrm{cond}})$, which is nothing but A^{cond}-modules as condensed abelian groups in $\mathrm{Mod}_A(D^{\mathrm{cond}}(\mathbb{Z}))$. Thus the assignment

$$\begin{aligned} \mathrm{Ring} &\to \mathrm{AnRing} \\ A &\mapsto (A^{\mathrm{cond}}, D^{\mathrm{cond}}(A)) \end{aligned}$$

induces a map between geometric set-ups $\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{AnStk}^{\mathrm{aff}}$. By Lemma 4.10, we induce a map $\widehat{\mathrm{Sh}}(\mathrm{Sch}^{\mathrm{aff}}) \to \mathrm{AnStk}$. In [6, Definition 3.18], Mathew defines a notion of descendable maps. By [6, Proposition 3.20], a map $f : X \to Y$ in $\mathrm{Sch}^{\mathrm{aff}}$ with respect to the standard six-functor formalism $D(\operatorname{Spec} A) = D(A)$ satisfies !-descent if and only if it is descendable. In other words, $\widehat{\mathrm{Sh}}(\mathrm{Sch}^{\mathrm{aff}}) = \mathrm{Stk}$, the classical notion of stacks with respect to the descendable topology.

[3] There are in fact size issues, which we skip for expository reasons. See [7, Warning 6.3.25] for details.

Definition 5.1. A finite family $\{\operatorname{Spec} B_i \to \operatorname{Spec} A\}$ is an ω_1-fpqc cover if B_i is an ω_1-compact A-algebra for all i and the induced map $A \to \prod B_i$ is faithfully flat.

Proposition 5.2 ([6, Proposition 3.31], [7, Proposition 6.4.3])**.** *If $\{\operatorname{Spec} B_i \to \operatorname{Spec} A\}$ is an ω_1-fpqc hypercover, then it is a descendable hypercover. In particular, there is a left exact colimit-preserving functor*

$$\mathrm{Stk}_{\omega_1-fpqc} \to \mathrm{AnStk}$$

where we use $\mathrm{Stk}_{\omega_1-fpqc}$ to denote the hypersheaves with respect to the ω_1-fpqc topology.

An important class of ω_1-fpqc stacks is given by the Betti stacks of anima: Consider the assignment

$$\mathrm{Prof} \to \mathrm{Sch}^{\mathrm{aff}} \tag{1}$$
$$S \mapsto \operatorname{Spec} \mathrm{Cont}(S, \mathbb{Z}). \tag{2}$$

Note that if $T \twoheadrightarrow S$ is a surjection, then the induced map $\mathrm{Cont}(S, \mathbb{Z}) \to \mathrm{Cont}(T, \mathbb{Z})$ is faithfully flat. One can see this by, for example, writing $S = \lim_{n \in \mathbb{N}^{op}} S_n$ as a sequential limit of finite sets S_n. But this implies that covers in Prof are sent to ω_1-fpqc covers, and we thus have a map

$$(-)_{\mathrm{Betti}} : \mathrm{CondAni} \to \mathrm{Stk}_{\omega_1-fpqc}.$$

Lemma 5.3. *For $S \in \mathrm{Prof}$, we have $D^{(\mathrm{cond})}(S_{\mathrm{Betti}}) = \mathrm{Sh}(S, D^{(\mathrm{cond})}(\mathbb{Z}))$.*

As mentioned in Example 1.6, there is a condensification functor Top $\to$ CondSet. For $X \in \mathrm{Top}$, we use the notation $X_{\mathrm{Betti}} = (\underline{X})_{\mathrm{Betti}}$.

Remark 5.4. This is where the definition of $\widehat{\mathrm{Sh}}$ comes into play; while we would like to define analytic stacks to be hypersheaves, it is not known if $(-)_{\mathrm{Betti}}$ lands in it.

To describe its quasi-coherent sheaves, we recall that $\mathrm{Sh}(S, D^{(\mathrm{cond})}(\mathbb{Z}))$ has an induced t-structure from $D^{(\mathrm{cond})}(\mathbb{Z})$ and we denote by

$$\widetilde{\mathrm{Sh}}(X, D(\mathbb{Z})) := \lim \left(\cdots \to \mathrm{Sh}(X, D^{\geq n}(\mathbb{Z})) \xrightarrow{\tau_{\geq n-1}} \mathrm{Sh}(X, D^{\geq n-1}(\mathbb{Z})) \to \cdots \right).$$

Proposition 5.5 ([4, 3.5.9], [8, II.1])**.** *If X is a locally compact Hausdorff space, then $D(X_{\mathrm{Betti}}) = \widetilde{\mathrm{Sh}}(X; D(\mathbb{Z}))$. When X is finite dimensional, then $\widetilde{\mathrm{Sh}}(X; D(\mathbb{Z})) = \mathrm{Sh}(X; D(\mathbb{Z}))$. Furthermore, the six functors coincide with the usual ones, which were briefly mentioned in Example 3.10.*

Our last example is the adic space.

Definition 5.6. A Huber pair (A, A^+) consists of the following data: A Banach algebra A which has an open subring A_0 whose subspace topology coincides with the I-adic topology for some finitely generated ideal $I \subseteq A_0$. Set A° to be the subring of power-bounded elements. This means that $a \in A^\circ$ if $\{a^n | n \in \mathbb{N}\}$ is bounded, which by definition means that, for any open set U of 0, there exists an open set V of 0 such that $\{a^n | n \in \mathbb{N}\} \cdot V \subseteq U$. An integrally closed subring $A^+ \subseteq A$ such that $A^+ \subseteq A^\circ$. A is said to be Tate if A contains a topologically nilpotent invertible element. In this case, (A, A^+) is said to be a Tate Huber pair. We use the notation $\mathrm{Ring}_{\mathbb{Z}}^{\mathrm{Hub}}$ to denote the category of Huber pairs.

Example 5.7. The pair $(\mathbb{Q}_p, \mathbb{Z}_p)$ is a Tate Huber pair. In fact, $\mathbb{Q}_p$ has a valuation given by

$$v_p\left(a_n p^n(1 + b_1 p + b_2 p^2 \cdots)\right) = n \in \mathbb{Z},$$

and $v_p(0) = \infty$. This valuation defines a Banach norm by $||x|| \coloneqq 2^{-v_p(x)}$. Open balls $B(0; r)$ gives a neighborhood basis of 0 and $\mathbb{Z}_p = B(0; 1)$.

Example 5.8. A slightly more advanced example of a Tate Huber pair is given by the Tate algebras $(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$. Here, for a non-archimedean Banach ring A with a valuation, the Tate algebra is defined to be the subring

$$A\langle T\rangle \coloneqq \{\sum a_n T^n \in A[[T]]\big| \lim_{n\to\infty} a_n = 0\}$$

of the power series ring consisting of those whose coefficients converge to 0. Then, $A\langle T\rangle$ is equipped with the Gauss norm

$$||\sum a_n T^n|| \coloneqq \sup |a_n|.$$

Again, $\mathbb{Z}_p\langle T\rangle$ is the unit ball inside $\mathbb{Q}_p\langle T\rangle$. One can get an even more advanced example of a Tate Huber pair, $(\mathbb{Q}_p^{cyc}\langle T^{1/p^\infty}\rangle, \mathbb{Z}_p^{cyc}\langle T^{1/p^\infty}\rangle)$, by adding all n-th roots of unity in $\mathbb{Q}_p$ and all p-th roots of T. To see it is a Huber pair, we remark that while $\mathbb{Q}_p^{cyc}$ doesn't admit a valuation, the norm naturally extends as a limit, since valuations extend to finite extensions. One can then define the norm on $\mathbb{Q}_p^{cyc}\langle T^{1/p^\infty}\rangle$ similarly and $\mathbb{Z}_p^{cyc}\langle T^{1/p^\infty}\rangle$ will again be the unit disk.

Let (A, A^+) be a Tate Huber pair. We will define an analytic ring $(A, A^+)_\square$. In order to do this, we first discuss the analytic ring $\mathbb{Z}[T]_\square$. Denote by $(\mathbb{Z}[T], \mathbb{Z})_\square$ the induced analytic ring $\mathbb{Z}[T]_{\mathbb{Z}_\square}$. One can compute that $(\mathbb{Z}[T], \mathbb{Z})_\square \otimes \mathbb{Z}[\hat{q}] = \mathbb{Z}[[q]][T]$ is the polynomial ring over the power series ring in the variable q.

Definition 5.9 ([7, Definition 5.2.1])**.** An object $M \in D\left((\mathbb{Z}[T], \mathbb{Z})_\square\right)$ is said to be T-solid if

$$1 - q : \mathrm{RHom}_{(\mathbb{Z}[T],\mathbb{Z})_\square}(\mathbb{Z}[[q]][T], M) \xrightarrow{\sim} \mathrm{RHom}_{(\mathbb{Z}[T],\mathbb{Z})_\square}(\mathbb{Z}[[q]][T], M)$$

is an isomorphism. We denote by $D(\mathbb{Z}[T]_\square)$ the subcategory of T-solid modules.

Roughly speaking, the abelian version would mean that M is T-solid if any null sequence $\{m_n\}$ is T-summable in that $\sum a_n T^n$ converges in M.

Definition 5.10 ([7, Definition 7.1.1])**.** For a Tate Huber pair, we define $D((A, A^+)_\square)$ to be the subcategory of $D(A)$ consisting of objects M such that M is $\mathbb{Z}$-solid and, for all $s \in A^+$, the $\mathbb{Z}[T]$-module M, given by $\mathbb{Z}[T] \to A^+$, $T \mapsto s$, is T-solid.

Proposition 5.11 ([1, Proposition 3.34])**.** *$(A, A^+)_\square$ is an analytic ring.*

For any Tate Huber pair and $f_1, \cdots, f_n, g$ such that they generate the unit ideal, one sets

$$A\langle\frac{f_1, \cdots, f_n}{g}\rangle \coloneqq A\langle T_1, \cdots, T_n\rangle \Big/ \overline{(gT_1 - f_1, \cdots, gT_n - f_n)}.$$

Here, $A\langle T_1, \cdots, T_n\rangle$ is the Tate algebra with n variables and we quotient as in the discrete situation to localize except we have to take the closure for the ideal. Furthermore, we use $A^+\langle\frac{f_1,\cdots,f_n}{g}\rangle$ to denote the integral closure of the image of $A^+\langle T_1, \cdots, T_n\rangle$.

Definition 5.12. We set $\mathrm{Aff}^{\mathrm{Hub}} := \mathrm{Ring}_{\mathbb{Z}}^{\mathrm{Hub},op}$ and use $\mathrm{Spa}(A, A^+)$ to denote the object corresponding to (A, A^+) and call it the affine adic space.

Similar to the case of schemes, one can make $\mathrm{Spa}(A, A^+)$ into a ringed space such that $\mathrm{Spa}(A\langle\frac{f_1,\cdots,f_n}{g}\rangle, A^+\langle\frac{f_1,\cdots,f_n}{g}\rangle)$ are open subspaces. Thus, we will call them rational subspaces. Since (A, A^+) is Tate, by [1, Proposition 4.3], there is a family of *nice* rational subspaces that form a neighborhood basis of $\mathrm{Spa}(A, A^+)$. However, the assignment given by

$$\mathrm{Spa}(A\langle\frac{f_1,\cdots,f_n}{g}\rangle, A^+\langle\frac{f_1,\cdots,f_n}{g}\rangle) \mapsto D((A, A^+)_\square)$$

doesn't always form a sheaf. A Tate Huber ring (A, A^+) is called sheafy if it does.

Definition 5.13. A Tate adic space is a ringed space $(X, \mathcal{O}_X)$ such that it is locally given by $\mathrm{Spa}(A, A^+)$ for a sheafy Tate Huber pair (A, A^+). We denote by TateAdic the category of Tate adic spaces.

Theorem 5.14 ([1, Theorem 4.1])**.** *Let X be a Tate adic space. The functor*

$$U \mapsto D((\mathcal{O}_X(U), \mathcal{O}_X^+(U)))$$

satisfies descent. As a result, there is a functor $\mathrm{TateAdic} \to \mathrm{AnStk}_{\mathbb{Z}_\square}$.

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