Riemann-Hilbert on contact manifolds

Christopher Kuo

Intro

The goal of this talk is to introduce a microlocal version of the Riemann-Hilbert correspondence. This classical correspondence establishes that, for a complex manifold X, the category of holomorphic vector bundles with connections $\mathrm{Vect}^{\nabla}(X)$ is equivalent to the purely topological category of local systems $\mathrm{Loc}(X)$.

Following the microlocal philosophy, this classical equivalence should be viewed as living on the zero section $X = 0_X$. We will introduce objects that generalize both sides, explain the correspondence, and discuss a microlocal version on the coprojective bundle $\mathbb{P}^*X$.

Our goal is to present a generalization to all complex contact manifolds V. If time permits, we will discuss an application with possible future directions toward geometric representation theory. This last part is joint work with Laurent Côté, David Nadler, and Vivek Shende.

Structure of the talk:

- Constructible sheaves
- D-modules
- Microlocalization
- To contact manifolds and an application

Constructible sheaves

Definition

A pre-sheaf F, valued in abelian groups (Ab), is a functor $F : \mathrm{Op}_X^{op} \to (Ab)$, meaning that there are assignments

$$U \mapsto F(U)$$
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Definition

A pre-sheaf F is a sheaf if its global data can be reconstructed (glued) from local pieces. More precisely, the sequence we have the following exact sequence:

$$0 \longrightarrow F(U) \longrightarrow \prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_{ij})$$

We denote the collection of sheaves by $\mathrm{Sh}(X; (\mathrm{Ab}))$ or simply $\mathrm{Sh}(X)$.

Constructible sheaves

The way to read the exact sequence
$0 \longrightarrow F(U) \longrightarrow \prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_{ij})$ is the following:

- Exactness at $F(U)$ means that two sections $s_1, s_2 \in F(U)$ are equal if their restrictions to each U_i agree.

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- Exactness at $F(U)$ means that two sections $s_1, s_2 \in F(U)$ are equal if their restrictions to each U_i agree.
- Exactness at $\prod_i F(U_i)$ means that a family of sections $\{s_i\}$ on the U_i glues to a section s on U if they agree on all double overlaps U_{ij}.

Constructible sheaves

Remark

For technical reasons, at some point during the talk, we will in fact need sheaves valued in $D(\mathbb{Z})$, chain complexes with quasi-isomorphisms inverted, (or in a suitable stable coefficient category). In other words, the target category should be an $(\infty, 1)$-category. A key difference from the ordinary case is that the gluing

$$F(U) \longrightarrow \lim \left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} F(U_{ijk}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

does not, in general, terminate after finitely many steps.

Constructible sheaves

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Here are some simple examples of sheaves

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- With adequate smooth structures, C^k, real analytic, or holomorphic functions form the sub-sheaves $\mathcal{C}^k_X$, $\mathcal{C}^\omega_X$, and $\mathcal{O}_X$.

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Definition

A sheaf $F \in \mathrm{Sh}(X)$ is said to be locally constant if there exists a cover $\{U_i\}$ of X such that $F|_{U_i}$ is constant, i.e., $F|_{U_i} \cong M_{U_i}$ for some abelian group M. The subcategory of locally constant sheaf is denoted by $\mathrm{Loc}(X)$.

Constructible sheaves

Remark

Loc(X) can equivalently be described as $\mathrm{Fun}\big(\pi_1(X), (\mathrm{Ab})\big)$, where $\pi_1(X)$ is the fundamental groupoid of X. In other words, an $L \in \mathrm{Loc}(X)$ corresponds to an assignment

$$(x \in X) \mapsto L_x,$$
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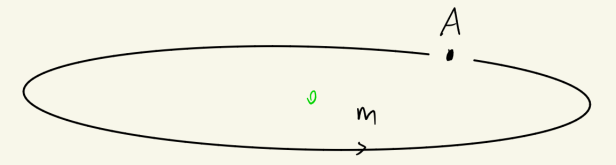
Example

On $\mathbb{C}^1 \setminus \{0\}$, a local system is equivalent to the data of an abelian group A together with a (monodromy) automorphism $m : A \xrightarrow{\cong} A$:



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Example

When $\mathcal{S} = \{X\}$ has only one stratum, then $\mathrm{Sh}_{\mathcal{S}}(X) = \mathrm{Loc}(X)$.

Constructible sheaves

Example (Extension by zero)

For topological space X and open $U \subseteq X$, define $\mathbb{Z}_U \in \mathrm{Sh}(X)$ by:

$$\mathbb{Z}_U(V) = \begin{cases} C^0(V; \mathbb{Z}) & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

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For closed $Z = X \setminus U$, define $\mathbb{Z}_Z$ by the exact sequence:

$$0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_Z \to 0$$

Both $\mathbb{Z}_U$ and $\mathbb{Z}_Z$ are constructible with respect to the stratification $\mathcal{S} = \{U, Z\}$.

Constructible sheaves

Another class of stratifications with an easy description of $\mathrm{Sh}_{\mathcal{S}}(X)$ occurs when all strata are contractible. Define a partial order on $\mathcal{S}$ by $\beta \leq \alpha$ if $X_\alpha \subseteq \overline{X_\beta}$.

Lemma

If all strata in $\mathcal{S}$ are contractible, then $\mathrm{Sh}_{\mathcal{S}}(X) = \mathrm{Fun}\left((\mathcal{S}, \leq)^{op}, (\mathrm{Ab})\right)$.

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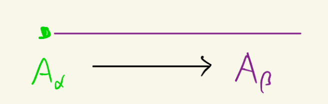
Example

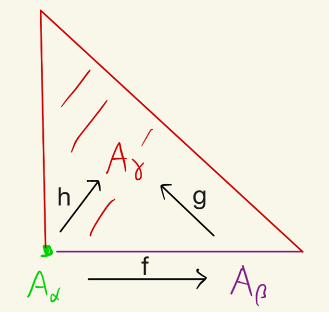
Consider the unit interval $\Delta^1 = [0,1]$ with the stratification $\mathcal{S} = \{\{0\}, (0,1]\}$. In this case, $(0,1] \leq \{0\}$ since $0 \in \overline{(0,1]}$, and $\mathrm{Sh}_{\mathcal{S}}(X)$ is given by representations of the quiver $\{\bullet \to \bullet\}$:



Constructible sheaves

Example

A similar example is given by the standard 2-simplex Δ^2 with strata $\{0\}$, $(0,1]$, and the remainder. In this case, an object has the following shape:



Here, the composition $g \circ f$ must agree with h.

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- If M is real analytic, define $\mathrm{Sh}_{\mathbb{R}\text{-}c}(M) := \bigcup_{\mathcal{S}} \mathrm{Sh}_{\mathcal{S}}(M)$, where $\mathcal{S}$ ranges over stratifications whose strata are locally closed subanalytic submanifolds.

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- If M is complex analytic, define $\mathrm{Sh}_{\mathbb{C}\text{-}c}(M) := \bigcup_{\mathcal{S}} \mathrm{Sh}_{\mathcal{S}}(M)$, where $\mathcal{S}$ ranges over stratifications whose strata are locally closed complex submanifolds.

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Proposition

With mild condition on M, any stratification $\mathcal{S}$ can be further refine to a triangulation $\mathcal{T}$. In this case, $\mathrm{Sh}_{\mathcal{S}}(M)$ can be viewed as a subcategory of $\mathrm{Fun}((\mathcal{T}, \leq), (\mathrm{Ab}))$.

D-modules

We now shift gears and let X be a complex manifold of dimension n. There is a sheaf of rings $\mathcal{D}_X$ on X whose sections are differential operators. Locally, in a coordinate system $\{z_i\}_{i=1}^n$, a section has the form

$$P = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \frac{\partial^\alpha}{\partial z^\alpha}, \quad c_\alpha \in \mathcal{O}_X, \quad c_\alpha = 0 \text{ for all but finitely many } \alpha.$$

Here, for a multi-index α, we use the notation

$$\frac{\partial^\alpha}{\partial z^\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}},$$

and the multiplication is given by composition.

D-modules

Definition

A D-module $\mathcal{M}$ is a sheaf of modules over $\mathcal{D}_X$, and the category of D-modules on X is denoted by $\mathcal{D}_X$-Mod.

Example

The sheaf $\mathcal{O}_X$ of holomorphic functions is canonically a D-module:

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More generally, $\mathcal{O}_X \otimes_{\mathbb{C}} L$ has a D-module structure for any finite rank $\mathbb{C}$-valued local system $L \in \mathrm{Loc}(X)^{\heartsuit}$.

Remark

In fact, a more invariant way to view $\mathcal{D}_X$ is to note that $\mathcal{O}_X$ and $\mathfrak{X}_X$, the sheaf of vector fields, can both be regarded as subsheaves of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$, via multiplication and differentiation, respectively. Then, $\mathcal{D}_X$ is the subring generated by $\mathcal{O}_X$ and $\mathfrak{X}_X$.

D-modules

Motivation: Study differential equations using homological algebra.

Construction:

- Let $P \in \mathcal{D}_X$ be a differential operator
- Define the quotient D-module $\mathcal{M}_P := \mathcal{D}_X / \mathcal{D}_X \cdot P$

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Key observation: Solutions correspond to morphisms:

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_P, \mathcal{O}_X)(U) = \{f \in \mathcal{O}_X(U) \mid Pf = 0\}$$

The sheaf-Hom gives precisely the solutions of the differential equation P.

D-modules

Definition

The **solution functor** $\mathsf{Sol} : \mathcal{D}_X\text{-Mod} \to \mathsf{Sh}(X; \mathsf{Vect}_{\mathbb{C}})$ is given by

$$\mathsf{Sol}(\mathcal{M}) := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

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Example (Complex line $\mathbb{C}^1$)

There is an exact sequence:

$$0 \to \mathcal{D}_{\mathbb{C}^1} \xrightarrow{\frac{\partial}{\partial z}} \mathcal{D}_{\mathbb{C}^1} \to \mathcal{O}_{\mathbb{C}^1} \to 0$$
$$P \mapsto P(1)$$

Computing solutions: $\mathrm{Sol}(\mathcal{O}_{\mathbb{C}^1})$ consists of $f \in \mathcal{O}_X$ with $\frac{\partial}{\partial z} f = 0$.
Therefore: $f = c$ for $c \in \mathbb{C}$, so $\mathrm{Sol}(\mathcal{O}_{\mathbb{C}^1}) = \mathbb{C}_{\mathbb{C}^1}$.

D-modules

Example (Ring of meromorphic functions at 0, $\mathcal{O}_{\mathbb{C}^1}(*0)$)

For meromorphic functions with a finite pole at 0, we have the resolution:

$$0 \to \mathcal{D}_{\mathbb{C}^1} \xrightarrow{\frac{\partial}{\partial z} z} \mathcal{D}_{\mathbb{C}^1} \to \mathcal{O}_{\mathbb{C}^1}(*0) \to 0$$

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Case analysis:

- If $0 \notin U$: solutions are $f = cz^{-1}$
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Key observation: Solution sheaves are *constructible* - this holds for a large class of D-modules.

Microlocalization

Topological setup: Let X be a topological space and $F \in \mathrm{Sh}(X)$.

Definition (Support of a sheaf)

The **support** of F is

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Enhancement: When M is a C^2-manifold, there is a more refined notion which we will define next.

Microlocalization

Approximate Definition (Part 1)

For $F \in \mathrm{Sh}(M; D(\mathbb{Z}))$, the **microsupport** $SS(F) \subseteq T^*M$ is a conic closed subset such that:

- $SS(F) \cap 0_M = \mathrm{supp}(F)$
- For $(x, \xi) \in T^*M \setminus 0_M$: $(x, \xi) \notin SS(F)$ if and only if locally testing on small open balls we have

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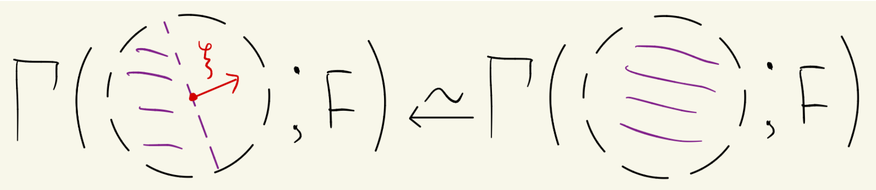

Geometric idea: ξ creates a directional barrier that divides regions.

Microlocalization

Approximate Definition (Part 2)

Continuing from the previous definition: $(x, \xi) \notin \mathrm{SS}(F)$ means that sections of F propagate uniquely across the barrier dictated by ξ.

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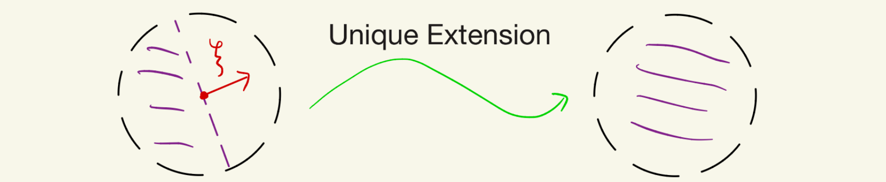

Key insight: ξ is not in the microsupport if sections extend uniquely across this directional barrier - no "obstruction" to propagation in the ξ direction.

Microlocalization

Example (Local systems have minimal microsupport)

If $L \in \mathrm{Loc}(M)$ is a local system, then $\mathrm{SS}(L) = 0_M$. (The converse is also true.)
Proof: Reduce to $L = \mathbb{Z}_M$ on small balls. Restriction from a ball to a half-ball is always the identity.

Example (Point singularity)

$\mathrm{SS}(\mathbb{C}_{\mathbb{C}^1 \setminus \{0\}}) = 0_{\mathbb{C}^1} \cup T_0^*\mathbb{C}^1$

Microlocalization

Example (Microsupport of extension by zero)

When M is a manifold and $U \subseteq M$ has smooth boundary:
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Proof idea: Let $x \in \partial U$ and ξ be an outward conormal at x.

- For ball B centered at x: $B \not\subseteq U$, so $\mathbb{Z}_U(B) = 0$
- For half-ball $B_\xi = B \cap \{z \mid \xi(z) < 0\}$ on the inward side: $B_\xi \subseteq U$, so $\mathbb{Z}_U(B_\xi) = \mathbb{Z}$

Microlocalization

Dual construction: For closed $Z = X \setminus U$, define $\mathbb{Z}_Z$ by the exact sequence: $0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_Z \to 0$

Example (Inward conormal)

When ∂U is smooth: $SS(\mathbb{Z}_{\overline{U}}) = N^*_{in}(U)$ the inward conormal bundle.

Pattern: Microsupport captures the "singular directions" - where sections cannot extend uniquely across barriers.

Microlocalization

Connection: Microsupport is closely related to symplectic geometry of T^*M.

Theorem (Kashiwara-Schapira)

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(1) Coisotropic property: $\mathrm{SS}(F)$ *is coisotropic. If* $\mathrm{SS}(F)$ *is stratified by locally closed submanifolds, then* $\mathrm{SS}(F)^{sm}$ *is a coisotropic submanifold.*

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(2) Constructibility characterization: *Assume* M *is real analytic and* $\mathrm{SS}(F)$ *is subanalytic. Then:* F *is constructible* $\Longleftrightarrow$ $\mathrm{SS}(F)$ *is Lagrangian*

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Theorem (Kashiwara-Schapira)

Let $F \in \mathrm{Sh}(M)$ be a sheaf.

(1) Coisotropic property: $\mathrm{SS}(F)$ *is coisotropic. If* $\mathrm{SS}(F)$ *is stratified by locally closed submanifolds, then* $\mathrm{SS}(F)^{sm}$ *is a coisotropic submanifold.*

(2) Constructibility characterization: *Assume M is real analytic and* $\mathrm{SS}(F)$ *is subanalytic. Then: F is constructible* $\iff$ $\mathrm{SS}(F)$ *is Lagrangian*

Key insight: Constructible sheaves correspond to Lagrangian microsupports - the smallest visible objects according to the uncertainty principle.

Microlocalization

Setup: Let $\pi : T^*X \to X$ be the projection and $\dot{T}^*X = T^*X \setminus 0_X$.

Microdifferential operators: There exists a ring $\mathcal{E}_{T^*X}$ of microdifferential operators - a complex conic sheaf on T^*X.

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Definition (Complex conic)

A sheaf $\mathcal{F}$ on T^*M is complex conic if $\mathcal{F}(\Omega) = \mathcal{F}(\mathbb{C}^\times \cdot \Omega)$ for open $\Omega \subseteq T^*M$.

In particular, $\mathcal{E}_{T^*X}|_{\dot{T}^*X}$ pulls back from a sheaf $\mathcal{E}_{\mathbb{P}^*X}$ on $\mathbb{P}^*X$.

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Next: We give the local form and structure of these operators.

Microlocalization

Local form: With coordinates $(\xi_1, \ldots, \xi_n)$ for covector directions, sections $P \in \mathcal{E}_{T^*X}$ are locally:

$$\sum_{\alpha \in \mathbb{Z}^n} c_\alpha \xi^\alpha, \quad c_\alpha \in \mathcal{O}_X, \quad c_\alpha = 0 \text{ when } \alpha \gg 0$$

Notation: Infinite terms allowed in negative directions, but bounded above.

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Composition rule: Extending symbol composition for differential operators:

$$P \circ Q = \sum_\alpha \frac{1}{\alpha!}\left(\frac{\partial^\alpha}{\partial \xi^\alpha} P\right)\left(\frac{\partial^\alpha}{\partial z^\alpha} Q\right)$$

Microlocalization

Example (Composition in $\mathcal{E}_{T^*\mathbb{C}^1}$)

Basic commutator: In $\mathcal{D}_{\mathbb{C}^1}$: $[\frac{\partial}{\partial z}, z] = 1$, i.e., $\frac{\partial}{\partial z} z = 1 + z\frac{\partial}{\partial z}$.
Viewing in $\mathcal{E}_{T^*\mathbb{C}^1}$:

Infinite series shows up naturally: Over $\{z \neq 0, \xi \neq 0\}$, consider z^{-1} and ξ^{-1}:

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$$\xi \circ z = \xi \cdot z + \frac{\partial}{\partial \xi}(\xi) \cdot \frac{\partial}{\partial z}(z) = \xi \cdot z + 1 = 1 + z\xi$$

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Infinite series shows up naturally: Over $\{z \neq 0, \xi \neq 0\}$, consider z^{-1} and ξ^{-1}:

- $z^{-1} \circ \xi^{-1} = z^{-1}\xi^{-1}$ (simple product)
- But $\xi^{-1} \circ z^{-1}$ gives an infinite series as below:

Example (Infinite series from composition)

As the $\circ$ is given by

$$P \circ Q = \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial \xi^\alpha} P \frac{\partial^\alpha}{\partial z^\alpha} Q,$$

computing $\xi^{-1} \circ z^{-1}$ results:

$$\xi^{-1} \circ z^{-1} =$$

Observation:

Microlocalization

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$$\begin{aligned}\xi^{-1} \circ z^{-1} &= \frac{1}{0!}\xi^{-1}z^{-1} + \frac{1}{1!}(\frac{\partial}{\partial \xi}\xi^{-1})(\frac{\partial}{\partial z}z^{-1}) \\ &\quad + \frac{1}{2!}(\frac{\partial}{\partial \xi}^2 \xi^{-1})(\frac{\partial}{\partial z}^2 z^{-1}) + \cdots \\ &= \xi^{-1}z^{-1} + \xi^{-2}z^{-2} + 2!\xi^{-3}z^{-3} + \cdots\end{aligned}$$

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Observation: Composition can produce infinite series even from simple rational functions.

Microlocalization

Definition (Microlocalization functor)

The microlocalization functor is given by

$$\begin{aligned} \mu : \mathcal{D}_X\text{-Mod} &\to \mathcal{E}_{T^*X}\text{-Mod} \\ \mathcal{M} &\mapsto \mathcal{E}_{T^*X} \otimes_{\pi^*\mathcal{D}_X} \pi^*\mathcal{M} \end{aligned}$$

The **characteristic variety** of a D-module $\mathcal{M}$ is

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Remark

This is equivalent to the usual definition via good filtrations.

Goal: Compute characteristic varieties using exact sequences.

Example (Computing characteristic varieties)

Microlocalization

Example (Computing characteristic varieties)

Case 1: For $\mathcal{O}_{\mathbb{C}^1}$, pullback the exact sequence:

$$0 \to \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \xrightarrow{\xi} \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \to \mu(\mathcal{O}_{\mathbb{C}^1}) \to 0$$

Since ξ is invertible away from the zero section: $\mathrm{Ch}(\mathcal{O}_{\mathbb{C}^1}) = 0_{\mathbb{C}^1}$.

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Case 2: For $\mathcal{O}_{\mathbb{C}^1}(*0)$:

$$0 \to \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \xrightarrow{\xi z} \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \to \mu(\mathcal{O}_{\mathbb{C}^1}(*0)) \to 0$$

Since ξz is invertible iff $\xi z \neq 0$: $\mathrm{Ch}(\mathcal{O}_{\mathbb{C}^1}(*0)) = 0_{\mathbb{C}^1} \cup T_0^*\mathbb{C}^1$.

Microlocalization

Key observation: The characteristic varieties we computed coincide with microsupports from earlier examples!

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Enhanced solution functor: Extend Sol to the derived category:

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\begin{aligned}
D^b_{coh}(\mathcal{D}_X) &\to \mathrm{Sh}(X; D(\mathbb{C})) \\
\mathcal{M} &\mapsto R\,\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)
\end{aligned}
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(Here $\mathrm{Sh}(X; D(\mathbb{Z}))$ means sheaves valued in chain complexes of $\mathbb{C}$-vector spaces with quasi-equivalence being inverted.)

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Theorem (Kashiwara-Schapira)

If $\mathcal{M} \in \mathcal{D}_X$-Mod is coherent, then: $\mathrm{SS}(\mathrm{Sol}(\mathcal{M})) = \mathrm{Ch}(\mathcal{M})$
In particular, if $\mathrm{Ch}(\mathcal{M})$ *is a complex Lagrangian, then* $\mathrm{Sol}(\mathcal{M}) \in \mathrm{Sh}_{\mathbb{C}\text{-}c}(X)$.

Microlocalization

Definition (Holonomic D-modules)

For coherent $\mathcal{M} \in \mathcal{D}_X$-Mod: $\mathcal{M}$ is **holonomic** if $\mathrm{Ch}(\mathcal{M})$ is Lagrangian.
For $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$: $\mathcal{M}$ is holonomic if all $H^k(\mathcal{M})$ are holonomic.

Theorem (Riemann-Hilbert correspondence (Kashiwara, Mebkhout))

The solution functor gives equivalences:

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Regular singularities: A holonomic D-module is **regular** if formal solutions converge. (The precise definition is technical.)

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- **Abelian level:**
$$\mathrm{Sol} : \mathcal{D}_X\text{-Mod}_{rh} \xrightarrow{\sim} \mathrm{Perv}(X)$$

where $\mathrm{Perv}(X)$ *is the category of perverse sheaves.*

Microlocalization

Goal: Define microlocalization for sheaves (topological side).

Difference: No ring to localize - instead, localize the category directly.

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Setup: For closed $X \subseteq T^*M$, define: $\mathrm{Sh}_X(M) \coloneqq \{F \mid \mathrm{SS}(F) \subseteq X\}$
Presheaf construction:

$$\begin{aligned} \mu\mathrm{sh}^{\mathrm{pre}} : \mathrm{Op}_{T^*M}^{op} &\to \mathbb{Z}\text{-Mod} \\ \Omega &\mapsto \mathrm{Sh}(M)/\,\mathrm{Sh}_{\Omega^{op}}(M) \end{aligned}$$

Intuition: $\mu\mathrm{sh}^{\mathrm{pre}}(\Omega)$ ignores differences outside Ω.

Definition (Microlocal sheaves)

The category-valued sheaf μsh on T^*M is the sheafification of μsh^{pre}. The inclusion $\dot{T}^*M \hookrightarrow T^*M$ induces the **microlocalization functor**:

$$\mathrm{Sh}(M) = \mu sh(T^*M) \to \mu sh(\dot{T}^*M)$$

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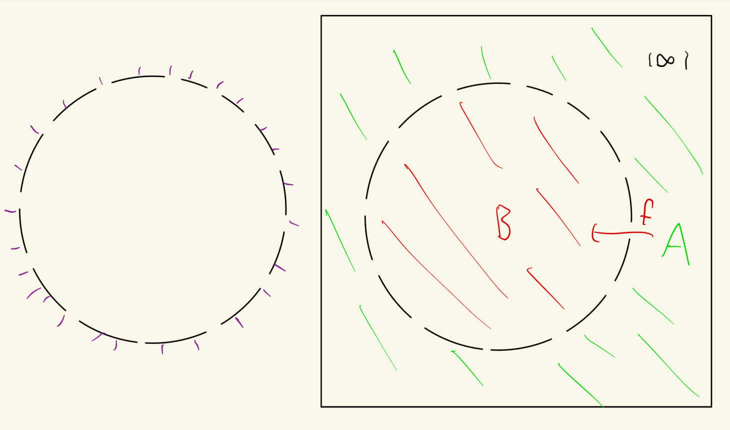
Key insight: This gives the topological analogue of D-module microlocalization - we can study sheaves "microlocally" by restricting to the cotangent bundle away from the zero section.

Example (Microlocalization on S^2)

Setup: $M = S^2 = \mathbb{R}^2 \cup \{\infty\}$ and $\Lambda = N^*_{out}(B_1(0))$.

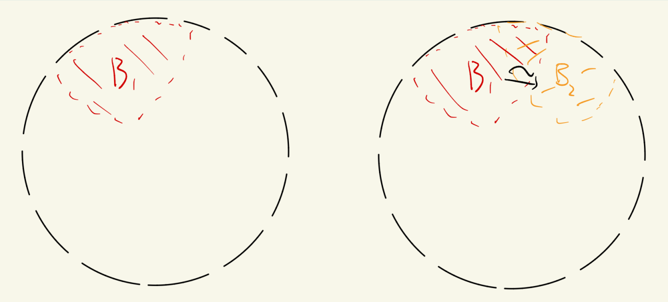


Observation: With only objects and 1-morphisms, it's the quiver $\{\bullet \to \bullet\}$.

Microlocalization

Example (Computing the microlocalization)

Note: Away from the zero section, $\dot{N}^*_{out}(B_1(0))$ is homotopic to S^1.



Result: $\mu sh_\Lambda(\dot{T}^*S^2) = \mathrm{Loc}(S^1)$ (local systems on S^1)

Microlocalization

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Summary:

- $\mathrm{Sh}_\Lambda(S^2)$ is approximately the quiver representation of $\{\bullet \to \bullet\}$.
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Summary:

- $\mathrm{Sh}_\Lambda(S^2)$ is approximately the quiver representation of $\{\bullet \to \bullet\}$.
- $\mu\mathrm{sh}_\Lambda(\dot{T}^*S^2) = \mathrm{Loc}(S^1)$ (local systems on S^1)

Microlocalization functor: $\mu : \mathrm{Sh}_\Lambda(S^2) \to \mathrm{Loc}(S^1)$ has image given by constant local systems on S^1.

Conclusion: μ is neither fully faithful nor essentially surjective:
For example, $\mathbb{Z}_{B_1(0)}$ is sent to $\mathbb{Z}_{S^1}$.

Microlocalization

Definition (Microlocal perverse sheaves)

For a complex manifold X, define $\mu\mathrm{Perv}(\dot{T}^*X)$ as the subcategory of $\mu\mathrm{sh}(\dot{T}^*X)$ which is locally in the image of:

$$\mathrm{Perv}(X) \hookrightarrow \mathrm{Sh}(X) \to \mu\mathrm{sh}(\dot{T}^*X)$$

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Remark

Since $\mu\mathrm{sh}$ / $\mu\mathrm{Perv}$ are $\mathbb{R}/\mathbb{C}$-conic, they pull back from sheaves on the cosphere/coprojective bundle, $S^*X/\mathbb{P}^*X$ (denoted by the same notation).

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Theorem (Microlocal Riemann-Hilbert (Andronikof, Waschkies))

The solution functor microlocalizes to an equivalence:

$$\mu\mathrm{Sol} : \mathcal{E}_X\text{-}\mathrm{Mod}_{rh} \xrightarrow{\sim} \mu\mathrm{Perv}(\mathbb{P}^*X)$$

To contact manifolds and an application

Goal: Globalize the microlocal Riemann-Hilbert correspondence.
(Joint work with Côté, Nadler, and Shende.)

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A complex contact manifold V is an odd-dimensional complex manifold with a maximally non-integrable hyperplane distribution $\mathcal{H} \subseteq TV$.

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Darboux theorem: Every point $p \in V$ has a neighborhood $\mathcal{U}$ with a contact embedding $g : \mathcal{U} \hookrightarrow \mathbb{P}^*X$.

To contact manifolds and an application

Kashiwara's construction: There exists a canonical sheaf of categories $\mathcal{E}_V\text{-}\mathcal{M}\mathrm{od}$ on V such that locally:

$$\mathcal{E}_V\text{-}\mathcal{M}\mathrm{od}\,|_{\mathcal{U}} \cong g^*\mathcal{E}_{\mathbb{P}^*X}\text{-}\mathcal{M}\mathrm{od}$$

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Significance: This provides a global Riemann-Hilbert correspondence on contact manifolds.

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Application setup: Let $\mathfrak{X}$ be an exact complex symplectic manifold with $\mathbb{C}^\times$-action of weight $k \in \mathbb{N}_+$.

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Local case: T^*X with coordinates (x, ξ) has the canonical Liouville form $\xi\, dx$ and the vector bundle structure provides a weight 1 action $t \cdot (x, \xi) = (x, t\xi)$.

To contact manifolds and an application

Motivation: $\mathcal{E}_{T^*X}$ is conic, so only captures conic symplectic data. For non-conic Lagrangians, we need WKB operators.

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WKB operators $\mathcal{W}_{T^*X}$**:** Linear over Laurent series $\mathbb{C}[[\hbar, \hbar^{-1}]]$:

$$P = \sum_{l \geq -m} f_l(z, w)\hbar^l, \quad f_l \in \mathcal{O}_{T^*X}$$

Embedding: $\mathcal{E}_{T^*X} \hookrightarrow \mathcal{W}_{T^*X}$ via $w = \hbar^{-1}\xi$.

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Generalization: Polesello-Schapira construct a canonical quantization $\mathcal{W}_{\mathfrak{X}}$-$\mathcal{M}od$ for any complex symplectic manifold $\mathfrak{X}$.

To contact manifolds and an application

Quantized action: The weight 1 action $f_t(z,\xi) = (z, t\xi)$ quantizes to a $\mathbb{C}[[\hbar, \hbar^{-1}]]$-linear automorphism F on $\mathcal{W}_{T^*X}$:

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Remark

The category of F-equivariant modules $(\mathcal{W}_{\mathfrak{X}}, F_{\mathfrak{X}})$-$\mathcal{M}$od often has geometric representation theory significance, e.g., Kashiwara-Rouquier's microlocalization of rational Cherednik algebras, or the category $\mathcal{O}$ of symplectic resolutions studied by Braden, Licata, Proudfoot, and Webster.

To contact manifolds and an application

Quantized action: The weight 1 action $f_t(z,\xi) = (z, t\xi)$ quantizes to a $\mathbb{C}[[\hbar, \hbar^{-1}]]$-linear automorphism F on $\mathcal{W}_{T^*X}$:

$$F_t(f(z,w)) = f(z,tw), \quad F_t(\hbar) = t\hbar$$

Remark

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Theorem (Petit)

There is an equivalence:

$$(\mathcal{W}_{\dot{T}^*X}, F)\text{-}\mathcal{M}od \simeq \mathcal{E}_{\mathbb{P}^*X}\text{-}\mathcal{M}od$$

To contact manifolds and an application

Theorem (Côté-K.-Nadler-Shende)

Let $\mathfrak{X}$ be an exact complex symplectic manifold with weight k action.

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Future direction:

To contact manifolds and an application

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Future direction: How much of the classical applications of perverse sheaves to geometric representation theory survives after microlocalization?