Riemann-Hilbert on contact manifolds

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Abstract

These are the notes for my talk, *Riemann-Hilbert on contact manifolds*, presented at the Max Planck Institute for Mathematics in August 2025. The goal of the talk is to introduce the Riemann-Hilbert correspondence from the basics to its microlocal version. Then we explain a joint work with Côté, Nadler, and Shende on gluing this microlocal version to a global Riemann-Hilbert on any given contact manifold, and an application to deformation quantization modules of symplectic resolutions.

The name Riemann-Hilbert correspondence obtains its name from the 21st Hilbert's problem: Proof of the existence of linear differential equations having a prescribed monodromy on a Riemann surface, and this correspondence is a vastly generalization of its answer. We begin the talk with recalling the definition of constructible sheaves, D-modules, and the solution functor relating them. The second step is microlocalization, which relies on the notion of microsupport of the constructible side and microdifferential operators on the D-modules side. In addition to providing a systematic way to understand the original correspondence, which should be thought of as living on the zero section, this notion also allows the correspondence itself to be microlocalized to the coprojective bundle. Lastly, we explain, in a joint work with Côté, Nadler, and Shende, how this microlocal version can be glued to a global microlocal Riemann-Hilbert on any given contact manifold, and how it can be applied to the study of deformation quantization modules of symplectic resolutions.

1 Constructible sheaves

For a topological space X, a sheaf $F \in \mathrm{Sh}(X)$ valued in abelian groups (Ab) is a functor

$$F : \mathrm{Op}_X^{op} \to (\mathrm{Ab}),$$

that is, an assignment

$$\begin{gathered} U \mapsto F(U), \\ (U \subseteq V) \mapsto (F(V) \to F(U)), \end{gathered}$$

such that global data can be reconstructed (glued) from local pieces. More precisely, if U admits an open cover $\{U_i\}_{i \in I}$, then the sequence

$$0 \longrightarrow F(U) \longrightarrow \prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_{ij})$$

is exact. Here, the first arrow is the product of the restriction maps, and the second arrow is the difference between the two ways of restricting

$$(U_i \supseteq U_{ij} \subseteq U_j, \quad U_{ij} := U_i \cap U_j).$$

Exactness at $F(U)$ means that two sections $s_1, s_2 \in F(U)$ are equal if their restrictions to each U_i agree for all $i \in I$. Likewise, exactness at $\prod_i F(U_i)$ means that a family of sections $\{s_i\}$ on the U_i glues to a section s on U if they agree on all double overlaps U_{ij}.

Remark 1.1. For technical reasons, at some point during the talk, we will in fact consider sheaves valued in chain complexes $\mathrm{D}(\mathbb{Z})$ (or in a suitable stable coefficient category). In other words, the target category should be an $(\infty, 1)$-category. A key difference from the ordinary case is that the gluing

$$F(U) \longrightarrow \lim \left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} F(U_{ijk}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

does not, in general, terminate after finitely many steps.

Example 1.2. The simplest example of a sheaf is the constant sheaf $\mathbb{Z}_X$, which is the sheafification of the presheaf $\mathbb{Z}_X^{\mathrm{pre}}$, where $\mathbb{Z}_X^{\mathrm{pre}}(U) = \mathbb{Z}$ and all restriction maps are given by $\mathrm{id}_{\mathbb{Z}}$. Sheafification is necessary because, for instance, for two disjoint open sets U and V, the equality

$$F(U \amalg V) = F(U) \oplus F(V)$$

must hold. In fact, one can identify $\mathbb{Z}_X(U)$ with the abelian group of locally constant functions from U to $\mathbb{Z}$, where $\mathbb{Z}$ is equipped with the discrete topology. Similarly, one can define M_X for any abelian group M by $M_X(U) := C^0(U; M)$ where M is equipped with the discrete topology.

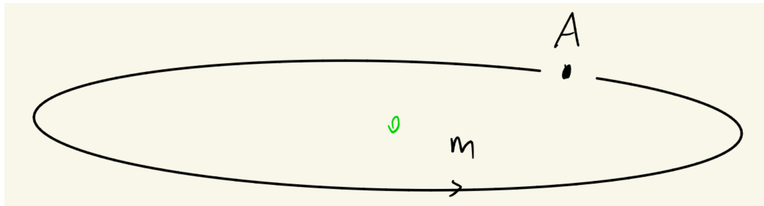
Definition 1.3. A sheaf $F \in \mathrm{Sh}(X)$ is said to be locally constant if there exists a cover $\{U_i\}$ of X such that $F|_{U_i}$ is constant. The subcategory of locally constant sheaf is denoted by $\mathrm{Loc}(X)$.

The category $\mathrm{Loc}(X)$ is in fact equivalent to $\pi_1(X, x_o)$-Mod or, without choosing a base point, $\mathrm{Fun}(\pi_1(X), (\mathrm{Ab}))$ where $\pi_1(X)$ is the fundamental groupoid of X. (In the higher categorical setting, one has to replace $\pi_1(X)$ by its higher categorical counterpart.) In other words, an object $L \in \mathrm{Loc}(X)$ is an assignment

$$\begin{aligned} (x \in X) &\mapsto L_x \\ \gamma : x \sim y &\mapsto L_x \xrightarrow{\cong} L_y. \end{aligned}$$

Note that this latter description implies that if X is contractible, then $\mathrm{Loc}(X) \cong (\mathrm{Ab})$ and all locally constant sheaves are in fact constant.

Example 1.4. On $\mathbb{C} \setminus \{0\}$, since the space is connected all stalk of a local system L is isomorphic to each other. As $\mathbb{C} \sim S^1$, the only other data needed is the monodromy m when going around the circle once:



Definition 1.5. Let $\mathcal{S} = \{X_\alpha\}_{\alpha \in I}$ be a stratification of X, i.e., $X = \amalg_{\alpha \in I} X_\alpha$ is a locally finite disjoint union of the strata X_α's. A sheaf F is said to be $\mathcal{S}$-constructible if $F|_{X_\alpha} \in \mathrm{Loc}(X_\alpha)$ for all $\alpha \in I$. We use $\mathrm{Sh}_{\mathcal{S}}(X)$ to denote the category of sheaves constructible with respect to $\mathcal{S}$

Example 1.6. When $\mathcal{S} = \{X\}$ has only one stratum, $\mathrm{Sh}_{\mathcal{S}}(X) = \mathrm{Loc}(X)$.

Example 1.7. Let $U \subseteq X$ be an open set. One can define $\mathbb{Z}_U \in \mathrm{Sh}(X)$ by

$$\mathbb{Z}_U(V) = \begin{cases} C^0(V;\mathbb{Z}), & \text{if } V \subseteq U, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, let $Z \coloneqq X \setminus U$ be the complement closed set. Then, the sheaf $\mathbb{Z}_Z$ is defined by the short exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_Z \to .0$$

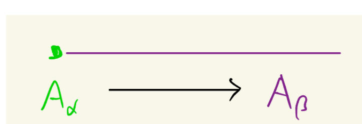
Take $\mathcal{S} = \{Z, U\}$. Then both $\mathbb{Z}_U$ and $\mathbb{Z}_X$ are $\mathcal{S}$-constructible. In fact, the category of $\mathcal{S}$-constructible sheaves fits in a recollement

$$\mathrm{Loc}(Z) \hookrightarrow \mathrm{Sh}_{\mathcal{S}}(X) \twoheadrightarrow \mathrm{Loc}(U).$$

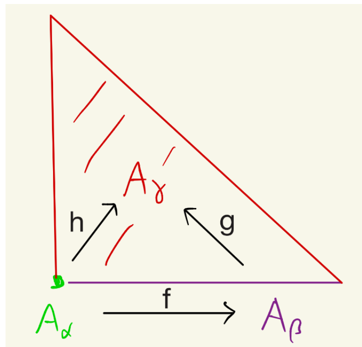
In general, $\mathrm{Sh}_{\mathcal{S}}(X)$ can be complicated. However, the situation is simply if $\mathcal{S}$ is a triangulation or, more generally, if all its strata are contractible, it admits a combinatorial representation: Notice that, for any $\mathcal{S}$, triangulation or otherwise, one can define an order "$\leq$" by $\beta \leq \alpha$ if $X_\alpha \subseteq \overline{X_\beta}$.

Lemma 1.8. *If all strata in $\mathcal{S}$ are contractible, then $\mathrm{Sh}_{\mathcal{S}}(X) = \mathrm{Fun}\,(\mathcal{S}, (\mathrm{Ab}))$ where we view $\mathcal{S}$ as a poset using the order defined above.*

Example 1.9. Consider the unit interval $\Delta^1 = [0,1]$ and the stratification $\mathcal{S} = \{\{0\}, (0,1]\}$. In this case, $(0,1] \leq \{0\}$ for $0 \in \overline{(0,1]}$ and $\mathrm{Sh}_{\mathcal{S}}(X)$ is given by representations of the quiver $\{\bullet \to \bullet\}$



Similarly, if we consider the standard 2-simplex Δ^2 by $\{0\}$, $(0,1]$, and the rest, an object will have the following shape:



Here, the composition $g \circ f$ has to agree with h.

Definition 1.10. Let M be a real analytic manifold. Then, define $\mathrm{Sh}_{\mathbb{R}\text{-}c}(M) := \cup_{\mathcal{S}} \mathrm{Sh}_{\mathcal{S}}(M)$ where $\mathcal{S}$ ranges over stratifications whose strata consist of locally closed subanalytic submanifolds. Similarly, if M is complex is complex analytic, define $\mathrm{Sh}_{\mathbb{C}\text{-}c}(M)$ similarly, replacing subanalytic closed submanifold by only complex analytic ones.

Remark 1.11. With some mild regularity conditions on M and $\mathcal{S}$, there always exists a triangulation $\mathcal{T}$ refining $\mathcal{S}$. Consequently, for any reasonable $\mathcal{S}$, the category $\mathrm{Sh}_{\mathcal{S}}(M)$ can be viewed as a subcategory of $\mathrm{Fun}((\mathcal{T}, \leq), (\mathrm{Ab}))$, and one can think of them as $\mathcal{T}$-representations with a collection of arrows being required to be invertible.

2 D-modules

Let X be a complex manifold of dimension n, there is a ring-valued sheaf $\mathcal{D}_X$, the ring of differential operators, whose sections are locally of the form

$$P = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \frac{\partial^\alpha}{\partial z^\alpha}, \; c_\alpha \in \mathcal{O}_X, \; c_\alpha = 0 \text{ for all but finitely many } \alpha.$$

Here, we choice a local coordinate x_i, and for each multi-index $\alpha \in \mathbb{N}^n$, we denote by

$$\frac{\partial^\alpha}{\partial z^\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$$

the linear operator given by composition of the standard ones. In other words, if we denote by $\mathcal{O}_X$ the ring of holomorphic functions, then $\mathcal{D}_X$ is the subring of $\mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{O})$ generated by $\mathcal{O}_X$ and derivation $\mathfrak{X}_X$.

Definition 2.1. A D-module $\mathcal{M}$ is a sheaf of modules over $\mathcal{D}_X$ and the category of D-modules on X is denoted as $\mathcal{D}_X\text{-Mod}$.

Example 2.2. The sheaf $\mathcal{O}_X$ of holomorphic functions is canonically a D-module. A holomorphic function $f \in \mathcal{O}_X \subseteq \mathcal{D}_X$ acts on $\mathcal{O}_X$ by multiplication and a vector field $v \in \mathfrak{X}_X \subseteq \mathcal{D}_X$ acts by differentiation. Leibniz's rule ensures that the action extends to the entire ring $\mathcal{D}_X$. More generally, if $L \in \mathrm{Loc}(X)^\heartsuit$ is a finite-rank $\mathbb{C}$-valued local system, then $\mathcal{O}_X \otimes_{\mathbb{C}} L$ has a D-module structure.

The originally motivation to study D-modules is to have a homological algebraic framework to study differential equations. Indeed, let $P \in \mathcal{D}_X$ be a differential operator and call the quotient D-module $\mathcal{M}_P := \mathcal{D}_X/\mathcal{D}_X \cdot P$. Then, one can compute that the sheaf-Hom

$$\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}_P, \mathcal{O}_X)(U) = \{f \in \mathcal{O}_X(U) | Pf = 0\}$$

is given by solution of P.

Definition 2.3. We denote by $\mathrm{Sol} : \mathcal{D}_X\text{-Mod} \to \mathrm{Sh}(X)^\heartsuit$ the functor given by $\mathrm{Sol}(\mathcal{M}) := \mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ and call it the solution functor.

Example 2.4. Consider the complex line $\mathbb{C}^1$ with a coordinate z. In this case, there is an exact sequence

$$0 \to \mathcal{D}_{\mathbb{C}^1} \xrightarrow{\frac{\partial}{\partial z}} \mathcal{D}_{\mathbb{C}^1} \to \mathcal{O}_{\mathbb{C}^1} \to 0;$$

in generally, $\mathcal{O}_X$ can be resolved by the Koszul complex. The discussion earlier then implies that $\mathrm{Sol}(\mathcal{O}_{\mathbb{C}^1})$ has sections $f \in \mathcal{O}_X$ such that $\frac{\partial}{\partial z} f = 0$, so $f = c$ for $c \in \mathbb{C}$. That is, $\mathrm{Sol}(\mathcal{O}_{\mathbb{C}^1}) = \mathbb{C}_{\mathbb{C}^1}$ from Example 1.2 (up to changing the coefficient to $\mathbb{C}$).

Example 2.5. A more interesting case is the ring of meromorphic functions with poles at 0, $\mathcal{O}_{\mathbb{C}^1}(*0)$. In this case, $\mathcal{O}_{\mathbb{C}^1}(*0)$ can be resolved by

$$0 \to \mathcal{D}_{\mathbb{C}^1} \xrightarrow{\frac{\partial}{\partial z} z} \mathcal{D}_{\mathbb{C}^1} \to \mathcal{O}_{\mathbb{C}^1} \to 0.$$

In other words, we are now solving $\frac{\partial}{\partial z}(zf) = 0$ so $zf = c$ for $c \in \mathbb{C}$. When $0 \notin U$, we have solution given by $f = cz^{-1}$. When $0 \in U$, by setting $z = 0$, we see that $c = 0$. Thus, $\mathrm{Sol}(\mathcal{O}_{\mathbb{C}^1}(*0)) = \mathbb{C}_{\mathbb{C}^1 \setminus \{0\}}$. Here, for an open set $U \subseteq X$, the sheaf $\mathbb{C}_U \in \mathrm{Sh}(X)$ is the extension by 0 of the constant sheaf $\mathbb{C}_U \in \mathrm{Sh}(U)$.

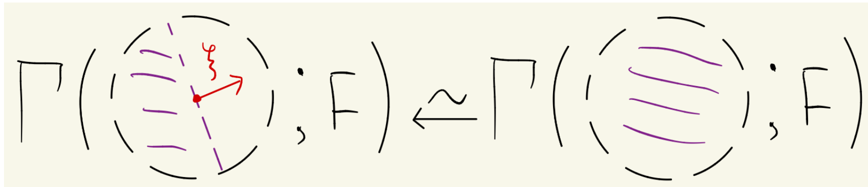
We notice that the solution sheaf of both examples are constructible. In fact, this holds true for a large class of D-modules. To have a systematic discussion, we introduce the microlocal notion.

3 Microlocalization

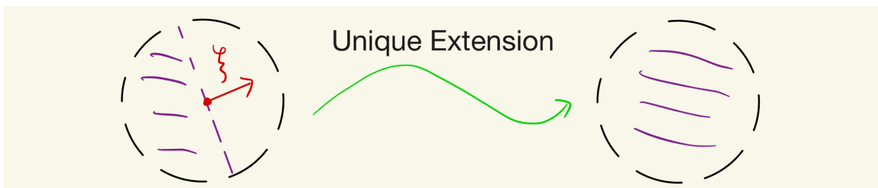
We begin with the topological side. Let X be a topological space. Recall that for a sheaf $F \in \mathrm{Sh}(X)$, there is a notion of the support $\mathrm{supp}(F) := \overline{\{x | F_x \neq 0\}}$. Equivalently, $\mathrm{supp}(F)^c$ is the largest open set U such that $F|_U = 0$. The support is not a very refine invariant. Since it cannot distinguish $\mathbb{Z}_U$ and $\mathbb{Z}_{\overline{U}}$. However, when M is a C^2-manifold, there is a enhancement: (The notion of microsupport works well only with the coefficient $\mathrm{D}(\mathbb{Z})$ and we hence replace (Ab) by it. Since $(\mathrm{Ab}) = \mathrm{D}(\mathbb{Z})^\heartsuit \subseteq \mathrm{D}(\mathbb{Z})$, we can embed the sheaves as well.)

Approximate Definition 3.1. For a $F \in \mathrm{Sh}(M; \mathrm{D}(\mathbb{Z}))$, its microsupport $\mathrm{SS}(F) \subseteq T^*M$ is a conic closed subset such that $\mathrm{SS}(F) \cap 0_M = \mathrm{supp}(F)$, and for $(x, \xi) \in T^*M \setminus 0_M$, we have

$(x, \xi) \notin \mathrm{SS}(F)$ if and only if, locally testing on small open balls,

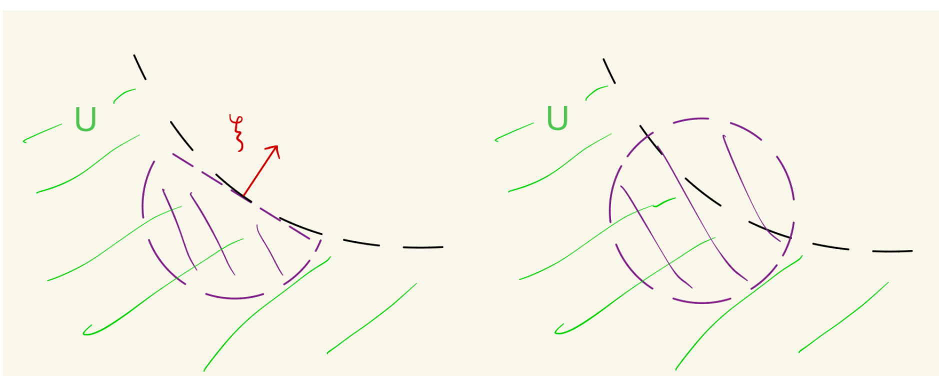


is an equivalence. Roughly speaking, since ξ divides a given open small open ball by half, this says that ξ is not in the microsupport if any section of F propagates uniquely across the barrier dictated by ξ,



Example 3.2. If $L \in \mathrm{Loc}(M)$ is a local system, then $\mathrm{SS}(L) = 0_M$. Indeed, the computation can be done on small balls so one can always assume $L = \mathbb{Z}_M$. But then, restriction from a ball to a half ball is always the identity.

Example 3.3. Let $U \subseteq X$ be an open set with a smooth boundary ∂U. We claim that $\mathrm{SS}(\mathbb{Z}_U) = N^*_{out}(U) := 0_U \cup N^*_{out}(\partial U)$. Here ∂U is a codimensional one submanifold so its conormal bundle is one dimensional and there is a well-defined notion of inward and outward. Let $(x, \xi) \in N^*_{out}(\partial U)$ such that $\xi \neq$. Let B be a small ball center at x. Shrinking B if needed, one can assume that the half ball $B_\xi := B \cap \{z | \xi(z) < 0\}$ is contained in U, as ξ points outward, so $\Gamma(B_\xi; \mathbb{Z}_U) = \mathbb{Z}$. On the other hand, B will always contain some point outside U so $\Gamma(B; \mathbb{Z}_U) = 0$, as $\mathbb{Z}_U$ is defined by extension by 0.



This same computation will show that $\mathrm{SS}(\mathbb{Z}_{\overline{U}}) = N^*_i n(U)$.

Example 3.4. On $\mathbb{C}^1$, a similar computation to the previous Example 3.3 will show that

$$\mathrm{SS}(\mathbb{C}_{\mathbb{C}^1 \setminus \{0\}}) = 0_{\mathbb{C}^1} \cup T^*_0 \mathbb{C}^1.$$

The notion of microsupport is closely related to symplectic geometry of T^*M.

Theorem 3.5 ([9, Theorem 6.5.4, Theorem 8.4.2]). *Let $F \in \mathrm{Sh}(M)$ be a sheaf.*

1. *The set $\mathrm{SS}(F)$ is coisotropic. If for simplicity we assume $\mathrm{SS}(F)$ is stratified by locally closed submanifolds, this means that the smooth locus $\mathrm{SS}(F)^{sm}$ is a coisotropic submanifold.*

2. *Assume further that M is real analytic and $\mathrm{SS}(F)$ is subanalytic. Then F is constructible if and only if $\mathrm{SS}(F)$ is Lagrangian.*

Similarly on the D-module side, we will define the characteristic variety $\mathrm{Ch}(\mathcal{M})$ for $\mathcal{M} \in \mathcal{D}_X$-Mod using microlocalization, which is equivalent to the usual definition.

Let $\pi : T^*X \to X$ is the projection and denote by $\dot{T}^*X := T^*X \setminus 0_X$ the open subset away from the zero section. There exists a ring of micro-differential operators $\mathcal{E}_{T^*X}$, which is a complex conic sheaf on T^*X. Here, "Complex conic" means that $\mathcal{E}_{T^*X}(\Omega) = \mathcal{E}_{T^*X}(\mathbb{C}^* \cdot \Omega)$ for $\Omega \subseteq T^*M$ open. In particular, $\mathcal{E}_{T^*X}|_{\dot{T}^*X}$ is the pullback of a sheaf $\mathcal{E}_{\mathbb{P}^*X}$ on $\mathbb{P}^*X$.

Let $(\xi_1, \cdots, \xi_n)$ be coordinates for the covector directions. A section $P \in \mathcal{E}_{T^*X}$ is locally of the form

$$\sum_{\alpha \in \mathbb{Z}^n} c_\alpha \xi^\alpha, \ c_\alpha \in \mathcal{O}_X, \ c_\alpha = 0 \text{ when} \, \alpha >> 0,$$

where the $c_\alpha \xi^\alpha$'s are viewed simply as functions (homogenous on the ξ's). That is, the summand can have infinitely many terms to the negative direction but has to be bounded above. In addition, some convergence conditions are required. This way, $\mathcal{E}_{T^*X}$ contains $\pi^*\mathcal{D}_X$ by viewing $\frac{\partial^\alpha}{\partial x^\alpha}$ as ξ^α. With this inclusion, the multiplication of $\mathcal{E}_{T^*X}$ is given by extending the symbol composition rules for differential operator:

$$P \circ Q = \sum_\alpha \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial \xi^\alpha} P \frac{\partial^\alpha}{\partial z^\alpha} Q.$$

Example 3.6. In $\mathcal{D}_{\mathbb{C}^1}$, z and $\frac{\partial}{\partial z}$ satisfies the equation $[\frac{\partial}{\partial z}, z] = 1$ or $\frac{\partial}{\partial z} z = 1 + z\frac{\partial}{\partial z}$. Viewed them as in $\mathcal{E}_{T^*\mathbb{C}^1}$, this is

$$\xi \circ x = \xi \cdot z + \frac{\partial}{\partial \xi}(\xi) \cdot \frac{\partial}{\partial z}(z) = \xi z + 1$$

where "$\cdot$" here is the simple multiplication between functions.

Over the open set $\{z \neq 0, \xi \neq 0\}$, there are more interesting sections z^{-1} and ξ^{-1}. The product $z^{-1} \circ \xi^{-1}$ is simply $z^{-1}\xi^{-1}$. However, when reversing the order, we obtain an infinite sum

$$\begin{aligned} \xi^{-1} \circ z^{-1} &= \frac{1}{0!}\xi^{-1}z^{-1} + \frac{1}{1!}(\frac{\partial}{\partial \xi}\xi^{-1})(\frac{\partial}{\partial z}z^{-1}) + \frac{1}{2!}\left((\frac{\partial^2}{\partial \xi^2}\xi^{-1})(\frac{\partial^2}{\partial z^2}z^{-1})\right) + \cdots \\ &= \xi^{-1}z^{-1} + \xi^{-2}z^{-2} + (2!)\xi^{-3}z^{-3} + \cdots . \end{aligned}$$

Definition 3.7. We define the microlocalization functor by

$$\begin{aligned} \mu : \mathcal{D}_X\text{-Mod} &\to \mathcal{E}_{T^*X}\text{-Mod} \\ \mathcal{M} &\mapsto \mathcal{E}_{T^*X} \otimes_{\pi^*\mathcal{D}_X} \pi^*\mathcal{M}. \end{aligned}$$

Then, for a D-module $\mathcal{M}$, the characteristic variety $\mathrm{Ch}(\mathcal{M}) := \mathrm{supp}(\mu(\mathcal{M}))$ is the support of its microlocalization.

Example 3.8. To find $\mathrm{Ch}(\mathcal{O}_{\mathbb{C}^1})$, we pullback and tensor the short exact sequence from Example 2.4 and obtain

$$0 \to \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \xrightarrow{\xi} \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \to \mu(\mathcal{O}_{\mathbb{C}^1}) \to 0.$$

Since ξ is invertible away from the zero section, $\mathrm{supp}(\mu(\mathcal{O}_{\mathbb{C}^1})) = 0_{\mathbb{C}^1} \subseteq T^*\mathbb{C}^1$.

Similarly, to find $\mathrm{Ch}(\mathcal{O}_{\mathbb{C}^1}(*0))$ from Example 2.5, one has to consider the exact sequence

$$0 \to \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \xrightarrow{\xi z} \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \to \mu\left(\mathcal{O}_{\mathbb{C}^1}(*0)\right) \to 0.$$

Since ξz is invertible if and only if $\xi z \neq 0$, we see that $\mathrm{Ch}(\mathcal{O}_{\mathbb{C}^1}(*0)) = 0_{\mathbb{C}^1} \cup T_0^*\mathbb{C}^1$.

We observe that the characteristic varieties of the above Example 3.7 and 3.8 coincide with the earlier microsupport examples we saw in Example 3.2 and 3.4. In fact, this phenomenon holds more generally. First, we derived Sol to the functor

$$\begin{aligned} D^b_{coh}(\mathcal{D}_X\text{-Mod}) &\to \mathrm{Sh}(X) \\ \mathcal{M} &\mapsto R\,\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \end{aligned}$$

but abuse the notation and still denote it by Sol. (Thus, $\mathrm{Sh}(X)$ has to mean the sheaves valued in chain complexes of $\mathbb{C}$-vector spaces.)

Theorem 3.9 ([9, Theorem 11.3.3])**.** *If $\mathcal{M} \in \mathcal{D}_X$-Mod in the abelian category is coherent, then*

$$\mathrm{SS}(\mathrm{Sol}(\mathcal{M})) = \mathrm{Ch}(\mathcal{M}).$$

In particular, if $\mathrm{Ch}(\mathcal{M})$ is a complex Lagrangian, then $\mathrm{Sol}(\mathcal{M}) \in \mathrm{Sh}_{\mathbb{C}\text{-}c}(X)$.

Definition 3.10. For a coherent $\mathcal{M} \in \mathcal{D}_X$-Mod, we say $\mathcal{M}$ is holonomic if $\mathrm{Ch}(\mathcal{M})$ is Lagrangian. For $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$, we say $\mathcal{M}$ is holonomic if all $H^k(\mathcal{M})$ is.

Before stating the Riemann-Hilbert correspondence, we need to mention there is a notion of regular singularity for holonomic D-modules. Since the definition is somewhat involved, we will be content with saying that a holonomic D-module is regular if formal solutions converge.

Theorem 3.11 ([6, 10])**.** *The solution functor* Sol *restricts to an equivalence*

$$\mathrm{Sol} : D^b_{rh}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Sh}_{\mathbb{C}\text{-}c}(X)^b.$$

Stated in the abelian level, there is an equivalence

$$\mathrm{Sol} : \mathcal{D}_X\text{-Mod}_{rh} \xrightarrow{\sim} \mathrm{Perv}(X).$$

Here, the target $\mathrm{Perv}(X)$ *is the category of perverse sheaves.*

We now introduce the corresponding notion of μ on the topology side. Since there isn't a ring to localize, we localize the category directly and first define a presheaf

$$\begin{aligned} \mu\mathrm{sh}^{\mathrm{pre}} : \mathrm{Op}^{op}_{T^*M} &\to \mathbb{Z}\text{-Mod} \\ \Omega &\mapsto \mathrm{Sh}(M)/\,\mathrm{Sh}_{\Omega^{op}}(M) \end{aligned}$$

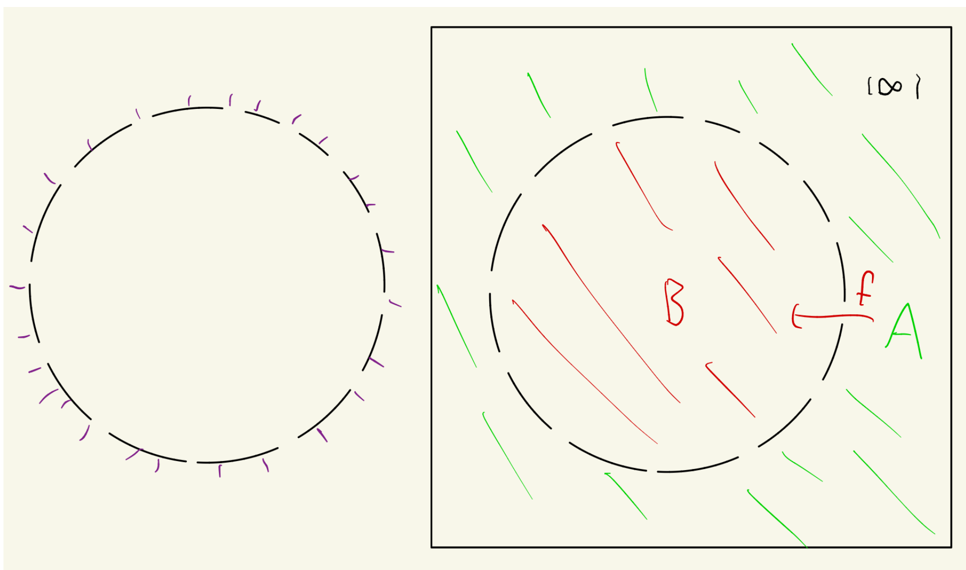
where for a closed subset $X \subseteq T^*M$, $\mathrm{Sh}_X(M) \coloneqq \{F \,|\, \mathrm{SS}(F) \subseteq X\}$. In other words, $\mu\mathrm{sh}^{\mathrm{pre}}(\Omega)$ ignore the difference outside Ω.

Definition 3.12. The category-valued sheaf $\mu\mathrm{sh}$ on T^*M is the sheafification of $\mu\mathrm{sh}^{\mathrm{pre}}$. The inclusion $\dot{T}^*M \hookrightarrow T^*M$, induces a microlocalization functor

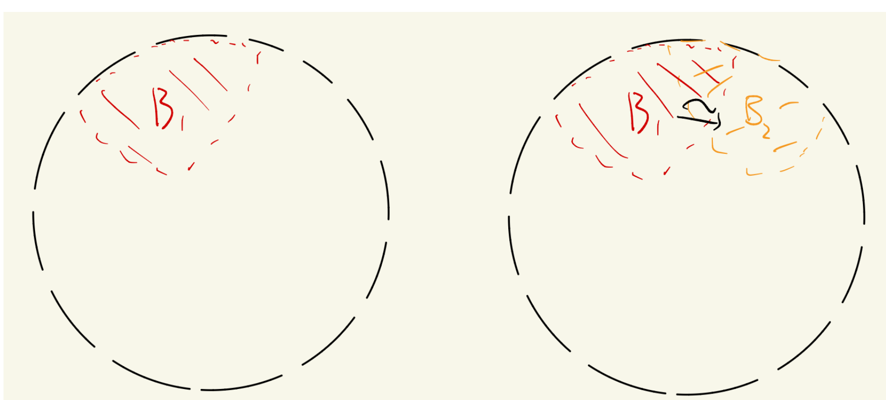
$$\mathrm{Sh}(M) = \mu\mathrm{sh}(T^*M) \to \mu\mathrm{sh}(\dot{T}^*M).$$

To get a feeling of what the above process does, we mention that there is a version $\mu\mathrm{sh}_\Lambda$ with a fixed microsupport condition $\Lambda \subseteq S^*M$.

Example 3.13. Let $M = S^2 = \mathbb{R}^2 \cup \{\infty\}$ and $\Lambda = N^*_{out}(B_1(0))$ be the outward conormal bundle of the unit open ball $B_1(0) = \{x^2+y^2 < 1\}$, i.e., inside $B_1(0)$, it's just the zero section and on the boundary, it's given by outward pointing covectors. And, similar to Example 3.4, considering only objects and 1-morphisms, the category is given by the quiver $\{\bullet \to \bullet\}$.



However, when computing $\mu\mathrm{sh}_\Lambda$ away from the zero section, since $\dot{N}^*_{out}(B_1(0))$ is homotopic to an S^1, the resulting category $\mu\mathrm{sh}_\Lambda(\dot{T}^*S^2) = \mathrm{Loc}(S^1)$, the same as local systems on S^1.



Thus, the microlocalization $\mu : \mathrm{Sh}_\Lambda(S^2) \to \mathrm{Loc}(S^1)$, which has image given by constant local systems, is neither fully-faithful nor essential surjective.

Definition 3.14. For a complex manifold X, we denote $\mu\mathrm{Perv}(\dot{T}^*X)$ the subcategory of $\mu\mathrm{sh}(\dot{T}^*X)$ which is locally in the image o the composition $\mathrm{Perv}(X) \hookrightarrow \mathrm{Sh}(X) \to \mu\mathrm{sh}(\dot{T}^*X)$.

Remark 3.15. As $\mu\mathrm{sh}/\mu\mathrm{Perv}$ are $\mathbb{R}/\mathbb{C}$-conic, it is the pullback of a sheaf from the cosphere bundle/coprojective and we will abuse and denote it by the same notation.

The Riemann-Hilbert correspondence admits a microlocalization.

Theorem 3.16 ([1, 13])**.** *The solution functor* Sol *microlocalizes to an equivalence*

$$\mu\mathrm{Sol} : \mathcal{E}_X\text{-}\mathrm{Mod}_{rh} \xrightarrow{\sim} \mu\mathrm{Perv}(\mathbb{P}^*X).$$

4 To contact manifolds

We close with a globalize version of the previous microlocal Riemann-Hilbert correspondence and an application. Recall that a complex contact manifold V is an odd dimensional complex manifold with a maximally non-integrable hyperplane distribution $\mathcal{H} \subseteq TV$.

Example 4.1. For a complex manifold X, the coprojective space $\mathbb{P}^*X$ has a canonical contact structure given by $\mathcal{H}_{(z,[\xi])} = \ker(\xi dz)$.

There is a Darboux theorem that for any point $p \in V$, there exists some open subset $\mathcal{U} \subseteq V$ and a contact embedding $g : \mathcal{U} \hookrightarrow \mathbb{P}^*X$ for some complex manifold X. Kashiwara shows in [7] that there exists a canonical sheaf of categories $\mathcal{E}_V\text{-}\mathcal{M}\mathrm{od}$ on V such that locally on a Darboux chart as above, $\mathcal{E}_V\text{-}\mathcal{M}\mathrm{od}\,|_{\mathcal{U}} \cong g^*\mathcal{E}_{\mathbb{P}^*X}\text{-}\mathcal{M}\mathrm{od}$. In a joint work with Côté, Nadler and Shende, we should that a similar object exists on the topology side and the microlocal Riemann-Hilbert correspondence in Theorem 3.16 glues.

Theorem 4.2 ([4, 5])**.** *Let V be a complex manifold.*

1. *There exists canonical sheaf of categories $\mu\mathrm{Perv}_V$, locally of the form $\mu\mathrm{Perv}_{\mathbb{P}^*X}$.*
2. *The equivalence $\mu\mathrm{Sol}$ from Theorem 3.16 glues to an equivalence*

$$\mu\mathrm{Sol} : \mathcal{E}_V\text{-}\mathrm{Mod}_{rh} \to \mu\mathrm{Perv}(V).$$

One application of the above theorem is the following: Let $\mathfrak{X}$ be an exact complex symplectic manifold with a $\mathbb{C}^\times$-action of weight k, $k \in \mathbb{N}_+$. That is, there is a Liouville 1-form α on $\mathfrak{X}$ such that $d\alpha$ is the symplectic form on $\mathfrak{X}$. Furthermore, there is a $\mathbb{C}^\times$-action f on $\mathfrak{X}$ such that $f_t^*\alpha = t^k\alpha$.

Such a set-up is of geometric representation interest, as it includes those studied in [8, 2, 3]. In particular, we will now consider the notion of W-modules, which in the case of a conical symplectic resolution, contained a generalized version of the category $\mathcal{O}$ as a subcategory. Recall that $\mathcal{E}_{T^*X}$ is a conic sheaf so it can only capture conic symplectic geometric data. To obtain information about possibly non-conic Lagrangian, one enlarges the ring to of WKB operators, $\mathcal{W}_{T^*X}$, linear over the Laurent series ring $\mathbb{C}[|\hbar,\hbar^{-1}]$ such that a section is of the form

$$P = \sum_{l \geq -m} f_l(z,w)\hbar^l, f_l \in \mathcal{O}_{T^*X}.$$

The ring $\mathcal{E}_{T^*X}$ embeds to $\mathcal{W}_{T^*X}$ by $w = \hbar^{-1}\xi$. Now, the canonical weight 1 action $f_t(z,\xi) = (z, t\xi)$ quantizes to an $\mathbb{C}[|\hbar, \hbar^{-1}]$-linear automorphism F on $\mathcal{W}_{T^*X}$ by $F_t(f(z,w)) = f(z,tw)$ and $F_t(\hbar) = t\hbar$, and this allows us to relate F-equivariant W-modules with E-modules.

Theorem 4.3 ([11]). *There is an equivalence*

$$(\mathcal{W}_{\dot{T}^*X}, F)\text{-Mod} = \mathcal{E}_{\mathbb{P}^*X}\text{-Mod}\,.$$

Similar to the case of E-modules, Schapira and Polesello shows in [12] that for any (possibly non-exact) complex symplectic manifold $\mathfrak{X}$, there exists a canonical $\mathcal{W}_{\mathfrak{X}}\text{-}\mathfrak{Mod}$. Combined with the above equivalence and the Riemann-Hilbert correspondence, we obtain

Theorem 4.4 ([5]). *Let $\mathfrak{X}$ be an exact complex symplectic manifold with a $\mathbb{C}^\times$-action of weight k. Then,*

1. *There exists a canonical action $F_{\mathfrak{X}}$ on $\mathcal{W}_{\mathfrak{X}}$.*
2. *There is an equivalence,*

$$(\mathcal{W}_{\mathfrak{X}}, F_{\mathfrak{X}})\text{-Mod}_{rh} = \mu\mathrm{Perv}_{\mathfrak{X}\times\mathbb{C}}(\mathfrak{X}\times\mathbb{C}).$$

Here, $\mathfrak{X}\times\mathbb{C}$ is the contactization of $\mathfrak{X}$.

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