Perverse microsheaves on contact manifolds

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Abstract

These are the notes for my talk, "Perverse microsheaves on contact manifolds," presented at EPFL at the Chair of Arithmetic Geometry in November 2025. The seminar is divided into two parts: a thirty-minute part for a general audience from the algebra group and a forty-five-minute part. For the first part, we will recall the classic Riemann-Hilbert correspondence and microlocalize the D-module side. For the second part, we will dedicate the discussion to the parallel construction on the constructible side, and explain a few technical subtleties.

1 Introduction

The notion of microlocalization has been long discovered for the theory of D-module [8, 1] and topological sheaf [11]. However, as one will see, the richer geometric given by the complex structure makes the theory of D-module quite straightforward. In particular, its microlocalization can be done directly at the level of rings using the so-called E-modules, by a very concrete, algebraic theory of pseudo-differential operators, going back all the way to the 70s [9]. On the contrary, the simplistic nature of constructible sheaves actually makes things complicated. For example, microlocalization has to be done in the level of derived category [10]. What's worse is that, while globalizing E-modules has been done by Kashiwara [7] in the 90s. The parallel theory wouldn't be accessible until Lurie provides a new foundation for derived categories about a decade ago [12, 13].

2 The Riemann-Hilbert correspondence

For a topological space X, a sheaf $F \in \mathrm{Sh}(X;\mathrm{Ab})$ valued in abelian groups (Ab) is a functor

$$F : \mathrm{Op}_X^{op} \to (\mathrm{Ab}),$$

that is, an assignment

$$U \mapsto F(U),$$
$$(U \subseteq V) \mapsto (F(V) \to F(U)),$$

such that global data can be reconstructed (glued) from local pieces. Let $\{U_i\}_{i\in I}$ be an open cover of U. The presheaf F is a sheaf if, for any two sections $s_1, s_2 \in F(U)$, we have $s_1 = s_2$ whenever $s_1|_{U_i} = s_2|_{U_i}$ in $F(U_i)$ for all $i \in I$, and, for any given family of sections $\{s_i\} \in F(U_i)$ that agree on all double overlaps U_{ij}, i.e., $s_i|_{U_{ji}} = s_j|_{U_{ij}}$, there exists $s \in F(U)$ such that $s|_{U_i} = s_i$.

Example 2.1. The presheaf of $\mathbb{C}$-valued functions ($U \mapsto \mathrm{Fun}(U, \mathbb{C})$) is a sheaf, since a function is determined by its values at each point. The sub-presheaf $\mathcal{C}^0_X : U \mapsto C^0(U)$ of continuous functions is also a sheaf, since continuity can be checked locally on opens. Similarly, if X is a smooth, real analytic, or complex manifold, then the sub-presheaf of smooth, real analytic, or holomorphic functions $\mathcal{C}^\infty_X/\mathcal{C}^\omega_X/\mathcal{O}_X$ is a sheaf, since these properties can be checked locally.

From now on, we assume X is a complex manifold of dimension n. There is a ring-valued sheaf $\mathcal{D}_X$, the ring of differential operators, whose sections are locally of the form

$$P = \sum_{\alpha\in\mathbb{N}^n} c_\alpha \frac{\partial^\alpha}{\partial z^\alpha},\ c_\alpha \in \mathcal{O}_X,\ c_\alpha = 0 \text{ for all but finitely many } \alpha.$$

Here, we choice a local coordinate x_i, and for each multi-index $\alpha \in \mathbb{N}^n$, we denote by

$$\frac{\partial^\alpha}{\partial z^\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$$

the linear operator given by composition of the standard ones. In other words, if we denote by $\mathcal{O}_X$ the ring of holomorphic functions, then $\mathcal{D}_X$ is the subring of $\mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{O})$ generated by $\mathcal{O}_X$ and derivation $\mathfrak{X}_X$.

Definition 2.2. A D-module $\mathcal{M}$ is a sheaf of modules over $\mathcal{D}_X$ and the category of D-modules on X is denoted as $\mathcal{D}_X\text{-Mod}$.

Example 2.3. The sheaf $\mathcal{O}_X$ of holomorphic functions is canonically a D-module. A holomorphic function $f \in \mathcal{O}_X \subseteq \mathcal{D}_X$ acts on $\mathcal{O}_X$ by multiplication and a vector field $v \in \mathfrak{X}_X \subseteq \mathcal{D}_X$ acts by differentiation. Leibniz's rule ensures that the action extends to the entire ring $\mathcal{D}_X$.

The originally motivation to study D-modules is to have a homological algebraic framework to study differential equations. Indeed, let $P \in \mathcal{D}_X$ be a differential operator and call the quotient D-module $\mathcal{M}_P := \mathcal{D}_X/\mathcal{D}_X \cdot P$. Then, one can compute that the sheaf-Hom

$$\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}_P, \mathcal{O}_X)(U) = \{f \in \mathcal{O}_X(U) | Pf = 0\}$$

is given by solution of P.

Definition 2.4. We denote by $\mathrm{Sol} : \mathcal{D}_X\text{-Mod} \to \mathrm{Sh}(X; \mathrm{Ab})$ the functor given by $\mathrm{Sol}(\mathcal{M}) := \mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ and call it the solution functor.

Example 2.5. Consider the complex line $\mathbb{C}^1$ with coordinate z. In this case, there is an exact sequence

$$0 \to \mathcal{D}_{\mathbb{C}^1} \xrightarrow{\frac{\partial}{\partial z}} \mathcal{D}_{\mathbb{C}^1} \to \mathcal{O}_{\mathbb{C}^1} \to 0;$$

in general, $\mathcal{O}_X$ can be resolved by the Koszul complex. The discussion above then implies that $\mathrm{Sol}(\mathcal{O}_{\mathbb{C}^1})$ has sections $f \in \mathcal{O}_X$ satisfying $\frac{\partial}{\partial z} f = 0$, hence $f = c$ for some $c \in \mathbb{C}$. This sheaf is usually denoted by $\mathbb{C}_{\mathbb{C}^1}$, the constant sheaf supported on $\mathbb{C}^1$.

Example 2.6. A more interesting case is the ring of meromorphic functions with finite poles at 0, $\mathcal{O}_{\mathbb{C}^1}(*0)$. In this case, $\mathcal{O}_{\mathbb{C}^1}(*0)$ can be resolved by

$$0 \to \mathcal{D}_{\mathbb{C}^1} \xrightarrow{\frac{\partial}{\partial z} z} \mathcal{D}_{\mathbb{C}^1} \to \mathcal{O}_{\mathbb{C}^1} \to 0.$$

In other words, we are now solving $\frac{\partial}{\partial z}(zf) = 0$ so $zf = c$ for $c \in \mathbb{C}$. When $0 \notin U$, we have solution given by $f = cz^{-1}$. When $0 \in U$, by setting $z = 0$, we see that $c = 0$. Thus, $\mathrm{Sol}(\mathcal{O}_{\mathbb{C}^1}(*0)) = \mathbb{C}_{\mathbb{C}^1 \setminus \{0\}}$.

For simplicity, when $U \subseteq X$ is an open set, the sheaf $\mathbb{C}_U$ is given by

$$\mathbb{C}_U(V) := \begin{cases} C^0(V, \mathbb{C}_{disc}), \ V \subseteq U \\ 0, \ V \subsetneq U, \end{cases}$$

where $C^0(V, \mathbb{C}_{disc})$ is the vector space of functions from V to $\mathbb{C}$ with the discrete topology. In other words, they are given by locally constant functions on V.

In fact, this functor identifies a large class of D-modules with a class of very simple sheaves. To see the full picture, we will need to derive the functor (but we will abuse notation and still write $\mathrm{Sol} : D^b(\mathcal{D}_X) \to \mathrm{Sh}(X; D^b(\mathbb{C}))$ for $\mathrm{Sol}(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))$.

Theorem 2.7 ([6, 14])**.** *The solution functor* Sol *restricts to an equivalence*

$$\mathrm{Sol} : D^b_{rh}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Sh}_{\mathbb{C}\text{-}c}(X)^b.$$

At the abelian level, there is an equivalence

$$\mathrm{Sol} : \mathcal{D}_X\text{-}\mathrm{Mod}_{rh} \xrightarrow{\sim} \mathrm{Perv}(X).$$

Here, 'rh' stands for regular holonomic, $\mathbb{C}$-c for complex constructible, and Perv for "perverse." We will conclude this discussion by explaining what "holonomic" means; "perversity" will be the focus of the next part: There is a notion of microlocalization, realized as the ring of micro-differential operators $\mathcal{E}_{T^*X}$, which is a complex conic sheaf on T^*X. Under local coordinates (z, ξ) as before, its sections have the form

$$\sum_{\alpha \in \mathbb{Z}^n} c_\alpha \xi^\alpha, \quad c_\alpha \in \mathcal{O}_X, \quad c_\alpha = 0 \text{ when } \alpha \gg 0,$$

where each $c_\alpha \xi^\alpha$ is viewed simply as a function (homogeneous in ξ). Let $\pi : T^*X \to X$ be the projection. There is a flat inclusion $\pi^*\mathcal{D}_X \hookrightarrow \mathcal{E}$, given by

$$\begin{aligned} \pi^*\mathcal{D}_X &\to \mathcal{E}, \\ f(z) &\mapsto f(z), \\ \frac{\partial^\alpha}{\partial z^\alpha} &\mapsto \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}, \end{aligned}$$

and the multiplication extends by the composition rule of principal symbols:

$$P \circ Q = \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial \xi^\alpha} P \frac{\partial^\alpha}{\partial z^\alpha} Q.$$

Note that for a sheaf on T^*X, one can restrict sections away from the zero section $\xi \neq 0$. In this case, for example, ξ_1 admits a "formal" inverse ξ_1^{-1}. Using $\mathcal{E}_X$, we can define the notion of the characteristic variety.

Definition 2.8. We define the microlocalization functor by

$$\begin{aligned} \mu : \mathcal{D}_X \text{-Mod} &\to \mathcal{E}_{T^*X} \text{-Mod}, \\ \mathcal{M} &\mapsto \mathcal{E}_{T^*X} \otimes_{\pi^*\mathcal{D}_X} \pi^*\mathcal{M}. \end{aligned}$$

Then, for a D-module $\mathcal{M}$, the characteristic variety $\mathrm{Ch}(\mathcal{M}) \coloneqq \mathrm{supp}(\mu(\mathcal{M}))$ is the support of its microlocalization.[1]

Example 2.9. To find $\mathrm{Ch}(\mathcal{O}_{\mathbb{C}^1})$, we pull back and tensor the short exact sequence from Example 2.5, obtaining

$$0 \to \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \xrightarrow{\xi} \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \to \mu(\mathcal{O}_{\mathbb{C}^1}) \to 0.$$

Since ξ is invertible away from the zero section, we have $\mathrm{supp}(\mu(\mathcal{O}_{\mathbb{C}^1})) = 0_{\mathbb{C}^1} \subseteq T^*\mathbb{C}^1$.

Similarly, to find $\mathrm{Ch}(\mathcal{O}_{\mathbb{C}^1}(*0))$ from Example 2.6, we consider the exact sequence

$$0 \to \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \xrightarrow{\xi \circ z} \mathcal{E}_{\mathbb{P}^*\mathbb{C}^1} \to \mu\left(\mathcal{O}_{\mathbb{C}^1}(*0)\right) \to 0.$$

Since $\xi \circ z$ is invertible if and only if $z\xi \neq 0$, we conclude that $\mathrm{Ch}(\mathcal{O}_{\mathbb{C}^1}(*0)) = 0_{\mathbb{C}^1} \cup T_0^*\mathbb{C}^1$.

Proposition-Definition 2.10 ([8, Theorem 2.15], [1, 1.6.18])**.** For a coherent $\mathcal{D}_X$-module, the characteristic variety $\mathrm{Ch}(\mathcal{M})$ is always coisotropic. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is said to be holonomic if $\mathrm{Ch}(\mathcal{M})$ is Lagrangian.

Remark 2.11. In fact, the existence of this formal inverse forces us to allow infinite series. For example, on the open set $\{z \neq 0, \xi \neq 0\} \subseteq T^*\mathbb{C}$, there are sections z^{-1} and ξ^{-1}. The product $z^{-1} \circ \xi^{-1}$ is simply $z^{-1}\xi^{-1}$. However, when the order is reversed, we obtain an infinite sum:

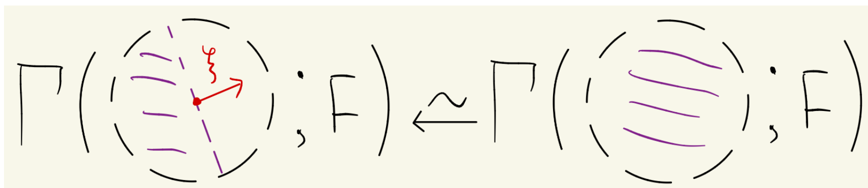
$$\begin{aligned} \xi^{-1} \circ z^{-1} &= \frac{1}{0!}\, \xi^{-1} z^{-1} + \frac{1}{1!} \left(\frac{\partial}{\partial \xi} \xi^{-1}\right)\left(\frac{\partial}{\partial z} z^{-1}\right) + \frac{1}{2!} \left(\left(\frac{\partial^2}{\partial \xi^2} \xi^{-1}\right)\left(\frac{\partial^2}{\partial z^2} z^{-1}\right) \right) + \cdots \\ &= \xi^{-1} z^{-1} + \xi^{-2} z^{-2} + (2!)\, \xi^{-3} z^{-3} + \cdots . \end{aligned}$$

[1] This is equivalent to the usual definition of the characteristic variety via good filtrations; see [8, Theorem 7.27] or [1, 8.2.12].

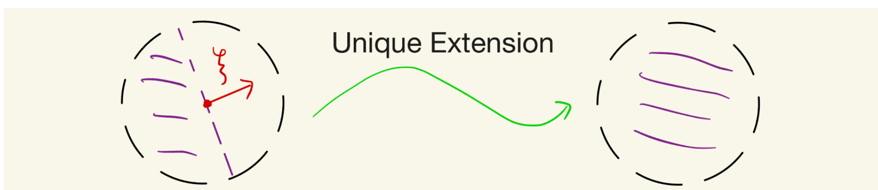
3 Perverse microsheaves on contact manifolds

We now turn to the discussion on the topological side. First, let M be a smooth manifold. For $F \in \mathrm{Sh}(M; D(\mathbb{Z}))$, there is a notion of microsupport $\mathrm{SS}(F) \subseteq T^*M$. On the zero section, $\mathrm{SS}(F) \cap 0_X = \mathrm{supp}(F)$ is the usual support. Away from the zero section, it is given by the (closure of the set of) co-directions of "non-propagation":

Approximate Definition 3.1. For $\xi \neq 0$, we say that $(x, \xi) \notin \mathrm{SS}(F)$ if and only if, when testing locally on small open balls,



is an equivalence. Roughly speaking, since ξ divides a small open ball into two halves, this means that ξ is not in the microsupport if every section of F propagates uniquely across the barrier determined by ξ:



Example 3.2. Let $x \in M$ be a point. Then, $F \in \mathrm{Sh}(M)$ is a local system near x if and only if $\mathrm{SS}(F) \subseteq 0_M$ near x.

We sometimes use the notation $\mathrm{SS}^\infty(F)$ to denote its projection to the sphere bundle $(\mathrm{SS}(F) \setminus 0_M)/\mathbb{R}^+$. In the case of complex manifold SS recovers the notion of Ch.

Proposition 3.3. *When $M = X$ is a complex manifold, we have $\mathrm{SS}(\mathrm{Sol}(\mathcal{M})) = \mathrm{Ch}(\mathcal{M})$ for any coherent $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_X)$.*

When we restrict our discussion to complex constructible sheaves $\mathrm{Sh}_{\mathbb{C}-c}(M)$, one can use the fiber[2] of the above map to measure the size of F at a smooth point $p \in \mathrm{SS}(F)^{sm}$, and get a notion of *microstalk*, which can in turn be used to define *perversity*. To give an easier definition[3], we will however first define the notion of microsheaves. For this part of the discussion, we go back to the general situation of a smooth manifold.

The microsupport satisfies the "triangle inequality": if $G \to F \to H$ is a fiber sequence in $\mathrm{Sh}(X; D(\mathbb{Z}))$[4], then we have $\mathrm{SS}(F) \subseteq \mathrm{SS}(G) \cup \mathrm{SS}(H)$. Thus, for a closed subset $C \subseteq T^*X$, the category

$$\mathrm{Sh}_C(X) := \{F \in \mathrm{Sh}(X; D(\mathbb{Z})) \mid \mathrm{SS}(F) \subseteq C\}$$

of sheaves microsupported in C is a stable subcategory of $\mathrm{Sh}(X)$.

[2] Or cone in the triangulated category language.

[3] See Remark 3.9 for the discussion on how to give the official definition for general real constructible sheaves and why it is complicated to do so.

[4] This is the way to say that it is an exact triangle in the higher categorical setting of stable categories.

Definition 3.4. The category-valued sheaf $\mu\mathrm{sh}_{T^*M}$ is the sheafification of the presheaf

$$\Omega \mapsto \mathrm{Sh}(M)/\,\mathrm{Sh}_{\Omega^c}(M).$$

That is, for an open set $\Omega \subseteq T^*M$, we discard the information outside Ω by taking the quotient. As SS is conic, the restriction $\mu\mathrm{sh}_{T^*M}|_{\dot{T}^*M}$ is the pullback of a unique sheaf on S^*M, denoted $\mu\mathrm{sh}_{S^*M}$. A section of μsh (in either case) is called a microsheaf.

For an open set $\Omega \subset T^*M$ and a microsheaf $\mathcal{F} \in \mu\mathrm{sh}(\Omega)$, we sometimes use the notation $\mathrm{SS}_\Omega(F) \subseteq \Omega$ to denote its support (or omit the Ω if it's clear from the context). This is because, for $\mathcal{G} \in \mu\mathrm{sh}^{\mathrm{pre}}(\Omega)$, the notion $\mathrm{SS}_\Omega(\mathcal{G}) \coloneqq \mathrm{SS}(G) \cap \Omega$, where $G \in \mathrm{Sh}(M)$ is any sheaf representing $\mathcal{G}$ in $\mu\mathrm{sh}^{\mathrm{pre}}(\Omega)$, is well-defined. And in this case, if we map G along the canonical map $\mu\mathrm{sh}^{\mathrm{pre}}(\Omega) \to \mu\mathrm{sh}(\Omega)$, then the two notions agree, as $(\mu\mathrm{sh}_{T^*M})_p = (\mu\mathrm{sh}^{\mathrm{pre}}_{T^*M})_p$ is given by germs of micro-presheaves.

Definition 3.5. For a closed set $C \subseteq T^*M$, we use the notation $\mu\mathrm{sh}_{T^*M,C}$ or simply $\mu\mathrm{sh}_C$ for the subsheaf of $\mu\mathrm{sh}_{T^*M}$ consisting of those microsheaves $\mathcal{F}$ which satisfy $\mathrm{SS}(\mathcal{F}) \subseteq C$. A similar definition can be made for $C \subseteq S^*M$.

Denote by $\pi : T^*M \to M$ the projection. Example 3.2 generalizes:

Proposition 3.6 ([11, Proposition 6.6.1])**.** *Let $p \in T^*M$ be a point and $N \subseteq M$ be a closed submanifold near $\pi(p)$. Then, any $\mathcal{F} \in (\mu\mathrm{sh}_{T^*_N M})_p$ is equivalent to A_N for some $A \in D(\mathbb{Z})$.*

Back to the complex situation, we can now define the microstalk μ_p at a complex smooth point $p \in \Lambda \coloneqq \mathrm{SS}(F)^{sm}$ such that $\Lambda \subseteq T^*X \xrightarrow{\pi} X$ is of constant projection rank. Recall that the latter map has projection rank $n-k$ if and only if $Y \coloneqq \pi(\Lambda)$ is a k-dimensional complex submanifold near $\pi(p)$.

Definition 3.7. In the above situation, the microstalk $\mu_p(F)$ is given by $A \in D(\mathbb{Z})$, where A is the unique object mapped to F in $(\mu\mathrm{sh}_{T^*_Y(X)})_p$ by the functor

$$\begin{aligned} D(\mathbb{Z}) &\to (\mu\mathrm{sh}_{T^*_Y(X)})_p \\ A &\mapsto A_Y[\dim_{\mathbb{C}} Y]. \end{aligned}$$

Note that we use Proposition 3.6 for the existence of A.[5]

We choose the degree shift for the definition of perversity.

Theorem-Definition 3.8 ([11, Definition 10.3.6, Theorem 10.3.12])**.** [6] For a complex constructible sheaf $F \in \mathrm{Sh}_{\mathbb{C}-c}(X)$, we use the notation $\mathrm{SS}(\mathcal{F})^{sm,cpr}$ to denote the set of smooth points in $\mathrm{SS}(F)$ such that the projection rank is locally constant. The perverse t-structure on $\mathrm{Sh}_{\mathbb{C}-c}(X)$ is given by the pair $(\mathrm{Sh}_{\mathbb{C}-c}(X)^{\le 0}, \mathrm{Sh}_{\mathbb{C}-c}(X)^{\ge 0})$ where

$$\mathrm{Sh}_{\mathbb{C}-c}(X)^{\le n} = \{F | \mu_p(F) \in D^{\le n}(\mathbb{Z}), p \in \mathrm{SS}(\mathcal{F})^{sm,cpr}\}$$
$$\mathrm{Sh}_{\mathbb{C}-c}(X)^{\ge n} = \{F | \mu_p(F) \in D^{\ge n}(\mathbb{Z}), p \in \mathrm{SS}(\mathcal{F})^{sm,cpr}\}$$

[5]We in fact need to use the notion of μhom from [11, Chapter 4] to show that this map is an equivalence. But we skip this discussion for simplicity.

[6]Again, this microlocal version is not the usual definition of perversity but can be shown to agree with it. In fact, the author does not know a direct proof that this pair defines a t-structure, as the book concludes it by comparing with the usual pair which is known to give a t-structure.

Remark 3.9. We remark on how to define the notion of microstalk for real constructible sheaves (on a real analytic manifold M) and why it is complicated to do so: For $F \in \mathrm{Sh}_{\mathbb{R}-c}(M)$ and $p = (x, \xi) \in \mathrm{SS}(M)^{sm}$, to get a notion $\mu_p(F)$, one has to choose a function φ such that $\varphi(x) = 0$ and $\mathrm{im}(d\varphi) \pitchfork \mathrm{SS}(F)$ at p; then [11, Proposition 7.5.3] implies that the object $\Gamma_{\{\varphi \geq 0\}}(F) \in D(\mathbb{Z})$ is well-defined up to a cohomological degree shift. In fact, the shift can be pinned down using the inertia index discussed in [11, Section A.3], and it gives the notion of "having type L at p" defined in [11, Definition 7.5.4]. To deduce the definition given in Definition 3.8, one might want to choose a contact transformation χ, explained in Proposition 3.10, to reduce to the case when $p \in \mathrm{SS}(M)^{sm,cpr}$. This wouldn't work in the general real case, though, as another degree shift depending on χ appears [11, Proposition 7.5.6]. The amazing fact about the complex situation is that the inertia index is always 0 [11, Exercise A.7] and so the ambiguity from the degree shift disappears.

Now we explain what contact transformations mean. For this part of the discussion, it's slightly bit more convenient to talk in the contact geometric setting. In this case, the behavior of F at $p \in \mathrm{SS}^\infty(F)^{sm}$ can be understood by choosing any $\tilde{p} \in \mathrm{SS}(F)$ that projections to p along $q : \dot{T}^*M \to S^*M$.

Proposition 3.10. *Let $\mathcal{U} \subseteq S^*M$ and $\mathcal{V} \subseteq S^*N$ be open subsets, and let $\chi : \mathcal{U} \xrightarrow{\sim} \mathcal{V}$ be a contactomorphism. Up to shrinking $\mathcal{U}$, there is an equivalence $\chi_* : \mu\mathrm{sh}_{\mathcal{U}} \cong \mu\mathrm{sh}_{\mathcal{V}}$.*

Recall that a real contact manifold $(V, \mathcal{H})$ is a pair consisting of a smooth manifold V (of real dimension $2n-1$) and a hyperplane distribution $\mathcal{H} \subseteq TV$ that is maximally non-integrable. Since $\mathcal{H} = \ker \alpha$ for some 1-form α, this is equivalent to the condition $\alpha \wedge (d\alpha)^{n-1} \neq 0$. A more intrinsic way to express this is to say that the sub-vector bundle $\hat{V} := (TV/\mathcal{H})^* \setminus 0_V \subseteq T^*V$ is a symplectic submanifold; in this case, $\hat{V}$ is called the symplectization of V. A contact manifold is said to be co-oriented if $\mathcal{H} = \ker \alpha$ globally. Equivalently, V is co-oriented if $\hat{V} \to V$ admits a section. We will assume all our real contact manifolds to be co-oriented for the rest of the discussion.

Given a contact manifold, one might ask whether we can glue μsh using Proposition 3.10. The answer is provided by Nadler and Shende in [16, 15] combined with Guillermou's concrete computation of the relevant map in [4]. [7] While what they give is a construction, in practice, it's often possible treating the machinery as a black box with a few properties.

To explain it, we first fix some notation. Denote by $U(n)$ the unitary groups inside n-by-n matrices. Recall that $BU(n-1)$ classifies rank $2n-2$ symplectic vector bundle and, since $\mathcal{H}$ is one by the fact that $d\alpha^{n-1}|_{\mathcal{H}} \neq 0$, it is classified by a map $\mathcal{H} : V \to B(n-1)$. There are also the stable version O and U of orthogonal and unitary group, and, by the Bott-periodicity, the quotient U/O is an infinite loop space (and so is a "commutative group" in an appropriate sense). Lastly, $\mathrm{Pic}(\mathbb{Z})$ is the Picard group of $D(\mathbb{Z})$.

Theorem 3.11. *Let V be a contact manifold, i.e., $\mathcal{H} = \ker \alpha$ globally. Then, the obstruction to glue μsh is given by the composition*

$$V \xrightarrow{\mathcal{H}} BU(n-1) \to BU \to B(U/O) \to \tau_{\leq 3} B(U/O) = B^2 \,\mathrm{Pic}(\mathbb{Z}). \tag{1}$$

[7] The theorem in fact holds for spectra-coefficient sheaves. We state the result here for simplicity. See [5] for details.

The meaning of this statement is that a choice of null-homotopy η for (1) defines a sheaf $\mu\mathrm{sh}_{V;\eta}$ and we term such a choice of null-homotopy a Maslov datum. Furthermore, the construction implies that if $h : \tau_1 = \tau_2$ is a homotopy between Maslov data induces an equivalence $h : \mu\mathrm{sh}_{V,\tau_1} = \mu\mathrm{sh}_{V,\tau_2}$.

To study the effect of higher homotopies, we need to understand the space of Maslov data: Let τ_1 and τ_2 be two Maslov data. The concatenation $\tau_1^{\mathrm{rev}}\#\tau_2$, where τ_1^{rev} means the reverse path, defines a loop

$$\tau_1^{\mathrm{rev}}\#\tau_2 \in \Omega_* \operatorname{Map}\left(V, B^2\operatorname{Pic}(\mathbb{Z})\right) = \operatorname{Map}\left(V, \Omega_* B^2\operatorname{Pic}(\mathbb{Z})\right) = \operatorname{Map}\left(V, B\operatorname{Pic}(\mathbb{Z})\right).$$

Lemma 3.12. *The space of Maslov data is a torsor for* $\operatorname{Map}(V, B\operatorname{Pic}(\mathbb{Z}))$. *In fact, concatenating with* τ_1^{rev}, $\tau_1^{\mathrm{rev}}\#(-)$, *identifies the space of null-homotopy of* $V \to B^2\operatorname{Pic}(\mathbb{Z})$ *with based loop space* $\Omega_*\operatorname{Map}(V, B^2\operatorname{Pic}(\mathbb{Z}))$.

Given two different null-homotopies $h, h' : \tau_1 = \tau_2$, we can similarly consider the concatenation

$$l = h^{\mathrm{rev}}\#h' \in \Omega_{\tau_1}\{\mathrm{null}\left(V \to B^2\operatorname{Pic}(\mathbb{Z})\right)\} = \Omega_*\operatorname{Map}(V; B\operatorname{Pic}(\mathbb{Z})) = \operatorname{Map}(V, \operatorname{Pic}(\mathbb{Z})).$$

But $\operatorname{Map}(V, \operatorname{Pic}(\mathbb{Z}))$ exactly classifies invertible local system L on V. We in fact have

Lemma 3.13. *Let* $L \in \operatorname{Loc}(V)$ *be the local system corresponding to* l *as above. Then, the two identification* $h_*, h'_* : \mu\mathrm{sh}_{V;\eta_1} = \mu\mathrm{sh}_{V;\eta_2}$ *is related by*

$$h'(-) = L \otimes h(-). \tag{2}$$

One way to choose the null-homotopy is by first choose a Maslov index, a null-homotopy of the further truncation

$$V \xrightarrow{\mathcal{H}} BU(n-1) \to BU \to B(U/O) \to \tau_{\leq 2}B(U/O) = B^2(\mathbb{Z}) = BU(1).$$

This is somewhat easier since the map $BU \to BU(1)$ is in fact given by the delooping of the map $\det^2 : U \to U(1)$, which is also known as twice the first Chern class $2c_1 \in H^2(BU;\mathbb{Z})$. As there is a fiber sequence in loop spaces,

$$B^3(\mathbb{Z}/2) \to B^2\operatorname{Pic}(\mathbb{Z}) \to B^2\mathbb{Z},$$

a null-homotopy of $m : 2c_1(\mathcal{H}) = 0$ corresponds to a map $m : V \to B^3(\mathbb{Z}/2)$. As it is classify by $[m] \in H^3(V, \mathbb{Z}/2)$, if $[m] = 0$, one can further choice a null-homotopy o, often referred as an orientation, to obtain a Maslov datum $\tau(m; o)$.

Another large class of Maslov data comes from polarizations, factorizations of $V \to BU$ to $V \to BO \to BU$, as $BO \to BU \to B(U/O)$ is canonically null-homotopic. For example, the vector bundle $\phi_M : S^*M \to BO(n)$ given by $\phi_M(x,\xi) = T_x^*M$ is a polarization for S^*M.

Proposition 3.14. *There is an equivalence* $\mu\mathrm{sh}_{S^*M;\phi_M} = \mu\mathrm{sh}_{S^*M}$ *where the right-hand side is the classical definition given in Definition 3.4.*

Since any contact manifold V is locally a of the form S^*M, and, as $B\operatorname{Pic}(\mathbb{Z})$ is connective, any Maslov data is locally homotopic to each other, we conclude that for $\mu\mathrm{sh}_{V;\tau} = f^* \mu\mathrm{sh}_{S^*M}$ for small enough $\mathcal{U} \subseteq V$ with an inclusion $f : \mathcal{U} \hookrightarrow S^*M$.

A variant we will use is the jet bundle $J^1M \coloneqq T^*M \times \mathbb{R}_s$ with the contact form $d\alpha_M + ds$, which has a similar polarization ϕ_M. As in Definition 3.5, for a closed set $C \subseteq V$, one denote by $\mu\mathrm{sh}_{V,C;\tau}$ the subsheaf consisting of objects supported in C. The following is a version of Proposition 3.6.

Corollary 3.15. *There is an equivalence $\mu\mathrm{sh}_{J^1M,0_M;\phi_M} = \mathrm{Loc}_M$ where the right hand side is the sheaf of (the category of) local systems on M.* [8]

We can now give an alternative definition of microstalk using Corollary 3.15. In fact, this definition works on general contact manifold V. We begin by fixing $\mathcal{F} \in \mu\mathrm{sh}_{V;\tau}(\mathcal{U})$ for some open set $\mathcal{U}$ and assume its support $\mathrm{SS}(\mathcal{F})$ is a Legendrian submanifold near p. In this case, there is the Weinstein neighborhood theorem saying that, on some tubular neighborhood $\mathcal{U}$ of L, $\mathcal{U} \cong J^1L$ (so there is a polarization ϕ_L on this neighborhood).

Definition 3.16. In the above situation, shrink $\mathcal{U}$ if needed, one can find a homotopy $h : \tau = \phi_L$. For example, when $\mathcal{U}$ is contractible, Lemma 3.12 implies that there exists $h_0 : \tau^{\mathrm{rev}} \# \phi_L = *$. With such a choice we obtain an identification

$$\mu\mathrm{sh}_{\mathcal{U},\tau} = \mu\mathrm{sh}_{J^1L,\phi_L} = \mathrm{Loc}_L.$$

Thus, there is a unique local system $A \in \mathrm{Loc}(L)$ that corresponds to $\mathcal{F}$ under the identification, and we define $\omega_p'(\mathcal{F}) = i_p^* A$ where $i_p : \{p\} \hookrightarrow L$ is the inclusion. In other words, we can define microstalk as simply taking the stalk after choosing a Weinstein coordinate.

In fact, the choice $h : \tau = \phi_L$ is what is usually referred as a secondary Maslov data, and we see in Lemma 3.13 that, in the simplest case when $\mathcal{U}$ is contractible, different choices could contribute to an object in $\operatorname{Pic}(\mathbb{Z})$, i.e., a cohomological degree shift. This explains why the inertia index mentioned in Remark 3.9 appears. More importantly, it explains why having a complex structure relies us from this ambiguity in Definition 3.7:

Assume the complex situation, i.e., $(V, \mathcal{H})$ is now complex. In this case, the symplectization $\hat{V} \to V$ is a $\mathbb{C}^\times$-bundle and admits a factorization

$$\hat{V} \to \hat{V}/\mathbb{R}^+ \xrightarrow{p} V$$

where p is a real contact manifold fibered by S^1 over V. The complex structure of $\mathcal{H}$ provides a canonical Maslov data η_{can} on $\hat{V}/\mathbb{R}^+$ and we can set $\mu\mathrm{sh}_V \coloneqq p_* \mu\mathrm{sh}_{hatV/\mathbb{R}^+:\eta_{can}}$. Similarly, for a complex Legendrian L, the fiber polarization $\phi_L : L \to BO(2n)$ in fact admits a lift to a complex fiber polarization $\phi_L : L \to BU(n) \to BO(2n)$. But this imply that η_{can} and ϕ_L must comes from the same Maslov index, discussed in the paragraph after Lemma 3.13, and we can thus chose h so that there is no shift.

Proposition 3.17. *Let X be a complex manifold and set $\omega_p(F) = \omega_p'(F)[-\dim_{\mathbb{C}} X]$. We have $\mu_p(F) = \omega_p(F)$.* [9]

[8] By (1), such an equivalence is a torsor for $H^0(M;\mathbb{Z})$. But we can once and for all fix one such that the equivalence respects inclusion on M.

[9] In fact, the comparison between these two definitions are realized by the Fourier-Sato transform.

The proposition shows that the alternative definition of microstalk

$$\omega_p(\mathcal{F}) := \omega_p'(\mathcal{F})[-\frac{1}{2}(\dim_{\mathbb{C}} V + 1)]$$

for any complex manifold V coincide with the classical one for the case when $V = \mathbb{P}^*X$. As t-structure can be checked locally, this allows us to globalizes it.

Proposition-Definition 3.18 ([2]). The pair $(\mu\mathrm{sh}^{\geq n}_{V,\mathbb{C}\text{-c}}, \mu\mathrm{sh}^{\leq n}_{V,\mathbb{C}\text{-c}})$ given by

$$\mu\mathrm{sh}^{\geq n}_{V,\mathbb{C}\text{-c}} = \{\mathcal{F} | \omega_p(\mathcal{F}) \in D^{\geq n}(\mathbb{Z}), p \in \mathrm{SS}(\mathcal{F})^{sm}\}$$

$$\mu\mathrm{sh}^{\leq n}_{V,\mathbb{C}\text{-c}} = \{\mathcal{F} | \omega_p(\mathcal{F}) \in D^{\leq n}(\mathbb{Z}), p \in \mathrm{SS}(\mathcal{F})^{sm}\}$$

define a microlocal perverse t-structure. We denote its heart by $\mu\mathrm{Perv}_V$ and call its objects the perverse microsheaves.

Theorem 3.19 ([3]). *There is an equivalence*

$$\mu\mathrm{Sol} : \mathcal{E}_V\text{-Mod}_{rh} \xrightarrow{\sim} \mu\mathrm{Perv}_V(V)$$

where $\mathcal{E}_V$ is the quantization of V defined by Kashiwara in [7].

We remark without spelling out the details that, to get invariants on a complex exact symplectic manifold $(\mathfrak{X}, \lambda)$, one will need a $\mathbb{C}^\times$-action of weight 1, i.e., $t^*\lambda = t\lambda$. Then, one sets $\mathrm{Perv}_{\mathfrak{X}}$ to be the subsheaf $\mathrm{Perv}_{\mathfrak{X}\times\mathbb{C}_s, \mathfrak{X}\times\{0\}}$ where $\mathfrak{X} \times \mathbb{C}$ is the contactization of $\mathfrak{X}$ with the contact form given by $\lambda + ds$.

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