A quick introduction of microlocal sheaf theory

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Abstract

This is the notes for my expository talk given in the Geometric Representation Theory seminar in SLMath in Spring 2024. The goal of this talk is to give a quick introduction to microlocal sheaf theory and its basic tool kit. We also mention a project of mine and a joint project with Wenyuan Li which further develop and utilize the machinery.

1 Motivation

Beginning with the pioneer work of Nadler-Zaslov [10, 9] and Tamarkin [11], microlocal sheaf theory has been applied to several field related to symplectic geometry. One of the recent theorem of Ganatra, Pardon, and Shende [2] proves that certain sheaf theoretic category in fact models the wrapped Fukaya category. Combining with the coherent-constructible correspondence, proposed by Fang, Liu, Treumann, and Zaslow [1], and finally proven by Kuwagaki [8], one obtains the following statement of toric mirror symmetry.

Theorem 1.1 ([2, Corollary 6.16, Example 7.25])**.** *Let Σ be a fan in $\mathbb{R}^n$. Denote by X_Σ the associated toric scheme and $i : \partial X_\Sigma \hookrightarrow X_\Sigma$ the inclusion of its toric boundary $\partial X_\Sigma := X_\Sigma \setminus (\mathbb{C}^*)^n$. Assume Σ is smooth and let $W_\Sigma : (\mathbb{C}^*)^n \to \mathbb{C}$ be the Hori-Vafa mirror potential. Then there is an equivalence*

$$\begin{array}{ccc} \mathrm{Coh}(\partial X_\Sigma) & \xrightarrow{\ i_*\ } & \mathrm{Coh}(X_\Sigma) \\ \| & & \| \\ \mathcal{W}\left(W_\Sigma^{-1}(\infty)\right) & \longrightarrow & \mathcal{W}\left((\mathbb{C}^*)^n, W_\Sigma^{-1}(\infty)\right) \end{array}$$

between the categories and functors, where the bottom inclusion is given by the fact that $W_\Sigma^{-1}(\infty) \hookrightarrow \mathbb{C}^*$ is a Liouville hyperplane.

To have a more natural statement in the microlocal sheaf theory frame work, let

$$\Lambda_\Sigma := \bigcup_{\sigma \in \Sigma} \sigma^\perp \times -\sigma \subseteq T^*T^n = T^n \times \mathbb{R}^n$$

the FLTZ skeleton at the infinity. We have the following equivalent expression:

$$\begin{array}{ccc} \mathrm{IndCoh}(\partial X_\Sigma) & \xrightarrow{\quad i^! \quad} & \mathrm{IndCoh}(X_\Sigma) \\ \Vert & & \Vert \\ \mathrm{Sh}_{\Lambda_\Sigma}(T^n) & \longrightarrow & \mu\mathrm{sh}_{\Lambda_\Sigma}(\Lambda_\Sigma) \end{array}$$

The goal of this talk is to introduce the standard toolkit in microlocal sheaf theory and, along the way, introduce the A-side categories and functors which show up in the second diagram.

2 Microlocal sheaf theory

2.1 Six functors

In this topological setting, we will assume our all spaces to be locally compact Hausdorff. We also fix a rigid symmetric monoidal (idempotent complete) small stable category $\mathcal{V}_0$, in the sense of [4], and we will use its Ind-completion $\mathcal{V} := \mathrm{Ind}(\mathcal{V}_0)$ as our coefficient. For this discussion, it is enough to take $\mathcal{V}_0 = \mathrm{Perf}\, k$ for some field k so $\mathcal{V} = k$-Mod. Let X be a space. We will consider the category of $\mathcal{V}$-valued sheaf $\mathrm{Sh}(X;\mathcal{V})$, which we will simply denote it as $\mathrm{Sh}(X)$ when it is unlikely to cause confusion. It is the full subcategory of $\mathrm{Fun}(Op_X^{op}, \mathcal{V})$ consisting of F such that, for any open U and any open cover $\{U_i\}$ of U, the canonical map built by the Čech nerve

$$F(U) \to \lim\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} F(U_{ijk}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots\right)$$

is an equivalence.

This assignment of $X \mapsto \mathrm{Sh}(X)$ admits the six-functor operations. That is, for a space X, there exists a symmetric monoidal product

$$(-) \otimes (-) : \mathrm{Sh}(X) \times \mathrm{Sh}(X) \to \mathrm{Sh}(X)$$

inherited from that of $\mathcal{V}$. For any $F \in \mathrm{Sh}(X)$, there is an adjunction $F \otimes (-) \vdash \mathcal{H}\mathrm{om}(F, -)$ and it provides an internal Hom

$$\mathcal{H}\mathrm{om} : \mathrm{Sh}(X)^{op} \times \mathrm{Sh}(X) \to \mathrm{Sh}(X).$$

For a map $f : X \to Y$, there is a $*$-adjunction, often referred as the "star"-adjunction,

$$f_* : \mathrm{Sh}(X) \rightleftharpoons \mathrm{Sh}(Y) : f^*$$

and a !-adjunction, often referred as the "shriek"-adjunction,

$$f^! : \mathrm{Sh}(Y) \rightleftharpoons \mathrm{Sh}(X) : f_!.$$

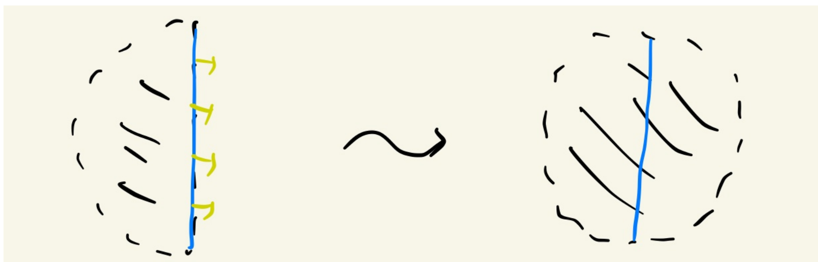
As usual, when f is proper $f_! = f_*$ and when f is smooth $f^! = f^* \otimes \omega_f$ where $\omega_f := f^! 1_N$. Other familiar properties are, for example, base change, the projection formula, etc..

2.2 Microsupport

Now we consider manifolds. For a sheaf $F \in \mathrm{Sh}(M)$, we first want to define an invariant, a conic closed subset $\mathrm{SS}(F) \subseteq T^*M$, generalizing the notion of support $\mathrm{supp}(F)$, which records the co-directions of non-propagation. Naively speaking, a point (x, ξ) is not in the microsupport $\mathrm{SS}(F)$ should mean that the sections can propagate toward ξ. Thus, in a coordinate, i fwe assume that $\xi = (1, 0, \cdots, 0)$, then this will mean that the restriction map

$$\Gamma(\mathbb{R}^n; F) \to \Gamma(\{x < 0\}; F)$$

is an equivalence.



Remark 2.1. The actual definition of $\mathrm{SS}(F)$ is more involved. Instead of checking at one point (x, ξ), one is required to find an open set $\Omega \ni (x, \xi)$ and check all points $(x', \xi') \in \Omega$. Furthermore, one cannot check on just one open neighborhood of x': Notice that there is a fiber sequence

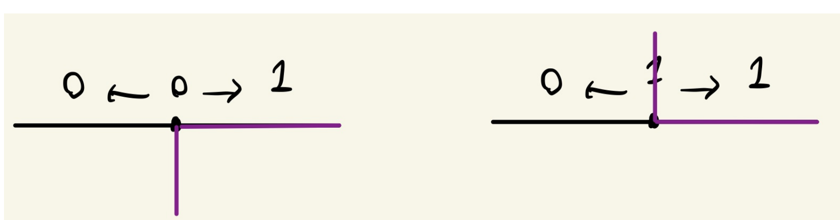
$$\Gamma_{\{x \geq 0\}}(\mathbb{R}^n; F) \to \Gamma(\mathbb{R}^n; F) \to \Gamma(\{x < 0\}; F)$$

where the left term is the sections supported on $\Gamma_{\{x \geq 0\}}(\mathbb{R}^n; F)$ and it vanishes if and only the right arrow is an equivalence. To have the official definition, one then needs to consider all functions ϕ defined near x' such that $d\phi_x = \xi'$ and check that the stalk

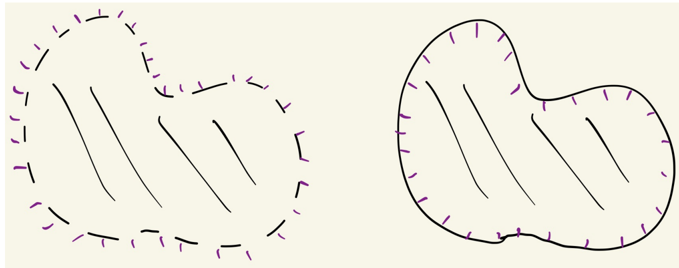
$$\left(\Gamma_{\{\phi \geq 0\}}(F)\right)_{x'} = 0$$

vanishes for all such ϕ.

Example 2.2. We have $\mathrm{SS}(1_{(0,\infty)}) = T^*_{0,\leq}\mathbb{R}^1 \cup [0, \infty)$ and $\mathrm{SS}(1_{[0,\infty)}) = T^*_{0,\geq}\mathbb{R}^1 \cup [0, \infty)$. More generally, for an open set $j : U \subseteq M$ with a smooth boundary ∂U, we have $\mathrm{SS}(j_! 1_U) = N^*_{out}(U) := N^*_{out}(\partial U) \cup 0_U$ and $\mathrm{SS}(j_* 1_U) = N^*_{in}(U)$. For example, $\Gamma((0,1); 1_{(0,\infty)}) = 1$ but $\Gamma((-1,1); 1_{(0,\infty)}) = 0$ so, at 0, the sections **do not** propagate toward the left and thus the negative co-direction **is** in the microsupport.



More generally, for a closed submanifold $Z \subseteq M$, we have $\mathrm{SS}(1_Z) = N^*(Z)$.



The above examples all have Lagrangian microsupport. But one can also have sheaves with strictly coisotropic microsupport

Example 2.3. We denote by $\mathcal{C}^\infty_M$ the sheaf of C^∞ functions. Then $\mathrm{SS}(\mathcal{C}^\infty_M) = T^*M$. Consider the real line $\mathbb{R}$, we have $\mathrm{SS}(\oplus_{x \geq 0} 1_x) = [0, \infty) \times \mathbb{R} \subseteq T^*\mathbb{R}$. Note that this microsupport has a non-empty boundary.

One main reason to consider microsupport is that it provides criterion for when canonical maps are equivalences. For example, one can consider the notion of non-characteristic, which provides the following generalization for the case when $f : Y \to X$ is smooth or the case when F is a local system.

Proposition 2.4 ([5, Proposition 5.4.13])**.** *Let $F \in \mathrm{Sh}(X)$ be a sheaf. If $f : Y \to X$ is non-characteristic to $\mathrm{SS}(F)$, then the canonical morphism $f^*F \otimes \omega_f \to f^!F$ is an isomorphism.*

To start making connection with symplectic geometry, we mention the following definition and theorem.

Definition 2.5. A stratification $\mathcal{S}$ is a locally finite decomposition $X = \amalg X_\alpha$ by locally closed submanifolds X_α. We always assume our stratifications to be Whitney. A sheaf $F \in \mathrm{Sh}(M)$ is called constructible if there exists a stratification $\mathcal{S}$ such that $F|_{X_\alpha} \in \mathrm{Loc}(X_\alpha)$. We use $\mathrm{Sh}_{\mathbb{R}-c}(M)$ to denote the subcategory consisting of constructible sheaves.

Theorem 2.6 ([5, Theorem 6.5.4, Proposition 8.3.10])**.**

1. *For any $F \in \mathrm{Sh}(M)$, the microsupport $\mathrm{SS}(F)$ is coisotropic.*
2. *Assume M is C^ω and $\mathrm{SS}(F)$ is subanalytic. Then $\mathrm{SS}(F)$ is Lagrangian if and only if F is constructible.*

Because of the above theorem, when talking about Lagrangians or Legendrians, we will assume them to be subanalytic (so the manifold M is C^ω). Fix a closed subset $X \subseteq T^*M$, we use the notation $\mathrm{Sh}_X(M)$ to denote

$$\mathrm{Sh}_X(M) := \{F \in \mathrm{Sh}(M) | \, \mathrm{SS}(F) \subseteq X\}$$

the category consisting of sheaves microsupported in X. Similarly, when we concern only the part of microsupport away from the zero section, we set

$$\mathrm{SS}^\infty(F) := (\mathrm{SS}(F) \setminus 0_M)/\mathbb{R}_{>0}) \,,$$

and, for a closed subset $X \subseteq S^*M$, we use a similar notation

$$\mathrm{Sh}_X(M) \coloneqq \{F \in \mathrm{Sh}(M) | \, \mathrm{SS}^\infty(F) \subseteq X\}$$

for the subcategory of sheaves microsupported (at the infinity) in X. That is, in this case, $\mathrm{Sh}_X(M) = \mathrm{Sh}_{(\mathbb{R}_{>0}X)\cup 0_M}(M)$. Let $\Lambda \subseteq S^*M$ be a Legendrian. The last theorem implies that $\mathrm{Sh}_\Lambda(M)$ consists of constructible sheaves. More detailed, one can always find a Whitney stratification $\mathcal{S}$ [5, Proposition 8.3.10] such that

$$\Lambda \subseteq N^*(\mathcal{S}) \coloneqq \bigcup_\alpha N^*(X_\alpha).$$

Proposition 2.7 ([2, Proposition 4.8]). *Let $\mathcal{S}$ a Whitney stratification and denote by $\mathrm{Sh}_{\mathcal{S}}(M)$ the subcategory sheaves constructible with respect to $\mathcal{S}$. Then we have $\mathrm{Sh}_{\mathcal{S}}(M) = \mathrm{Sh}_{N^*\mathcal{S}}(M)$.*

Corollary 2.8. *For a Legendrian Λ, the category $\mathrm{Sh}_\Lambda(M)$ is compactly generated. Moreover, corepresentatives of stalks and microstalks, functors of the form $F \mapsto \mu_{(x,\xi)}(F) \coloneqq \left(\Gamma_{\{\phi \geq 0\}}(F)\right)_x$ in the sense of Remark 2.1, form a generating set.*

Example 2.9. Consider the case when $M = S^1$ and $\Lambda = T^*_{0,<}S^1 = \{(0,-1)\}$. The data to decide a sheaf $F \in \mathrm{Sh}_\Lambda(S^1)$ consists of the stalk A and a possibly non-invertible endomorphism $\alpha : A \to A$ when restricting to the to the right. For example, denote by $\pi : \mathbb{R}^1 \to S^1$ the projection, then $\pi_! 1_{(0,\infty)}$ is such a sheaf. For such a sheaf F, up to a shift [5, Proposition 7.5.3], $\mu_{(x,\xi)}(F) = \mathrm{fib}(\alpha : A \to A)$ so F is a local system if $\mu_{(x,\xi)}(F) = 0$. Clearly, $F = 0$ if and only if $A = 0$.

2.3 Microsheaves

Up until now, we've mostly working on the base manifold and use $\mathrm{SS}(F)$ as an auxiliary tool. The following construction will allow us to work directly on the cotangent bundle T^*M.

Definition 2.10. We define the conic sheaf $\mu\mathrm{sh}_{T^*M}$ to be the sheafification of the presheaf

$$\begin{aligned} \mu\mathrm{sh}^{\mathrm{pre}}_{T^*M} : Op^{op}_{T^*M} &\to \mathrm{st} \\ \Omega &\mapsto \mathrm{Sh}(M)/\,\mathrm{Sh}_{T^*M\setminus\Omega}(M). \end{aligned}$$

Here, we use st to denote the category of stable categories. Denote by $p : \dot{T}^*M \to S^*M$ the projection to the cosphere bundle. Because $\mu\mathrm{sh}_{T^*M}$ is conic, on $\dot{T}^*M$, it is a pullback $\mu\mathrm{sh}_{S^*M}$ from S^*M or $\mu\mathrm{sh}_{T^*M}|_{\dot{T}^*M} = p^*\,\mu\mathrm{sh}_{S^*M}$. We refer the objects of either $\mu\mathrm{sh}_{T^*M}(\Omega)$ or $\mu\mathrm{sh}_{S^*M}(\Omega)$ as microsheaves on Ω. When the context is clear, one often surpass the notation and simply write $\mu\mathrm{sh}$ for either case.

For $F \in \mu\mathrm{sh}^{\mathrm{pre}}(\Omega)$, there is a notion of $\mathrm{SS}_\Omega(F) \coloneqq \mathrm{SS}(\tilde{F}) \cap \Omega$ for any representative $\tilde{F}$. This follows from the triangle inequality of microsupport: If $F \to G \to H$ is a fiber sequence, then

$$((\mathrm{SS}(F) \setminus \mathrm{SS}(H)) \cup (\mathrm{SS}(H) \setminus \mathrm{SS}(F))) \subseteq \mathrm{SS}(G) \subseteq (\mathrm{SS}(F) \cup \mathrm{SS}(H))\,.$$

The notion further descends to $\mu\mathrm{sh}$ since $\mu\mathrm{sh}_{(x,\xi)} = \operatorname*{colim}_{\Omega \ni (x,\xi)} \mu\mathrm{sh}^{\mathrm{pre}}(\Omega)$ is computed by germs. (This depends on the subtle fact that the coefficient is st.)

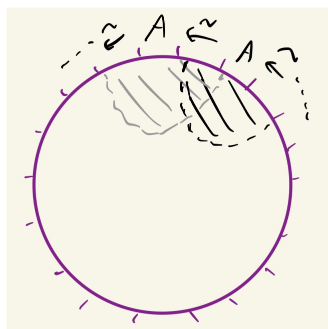
Definition 2.11. For a closed subset X of S^*M or a conic closed subset of T^*M, we use the notation $\mu\mathrm{sh}_{T^*M;X}$ or, when the context is clear, $\mu\mathrm{sh}_X$ to denote the subsheaf of $\mu\mathrm{sh}_{S^*M}$ / $\mu\mathrm{sh}_{T^*M}$ consisting of objects microsupported in X (at the infinity).

Lemma 2.12. *For a Legendrian $\Lambda \subseteq S^*M$ and an open $\Omega \subseteq S^*M$, the category $\mu\mathrm{sh}_\Lambda(\Omega)$ is compactly generated. In fact, in generic posistion, for Ω small enough, $\mu\mathrm{sh}_\Lambda(\Omega)$ is a quotient*

$$\mu\mathrm{sh}_\Lambda(\Omega) = \mu\mathrm{sh}_\Lambda^{\mathrm{pre}}(\Omega) := \mathrm{Sh}_\Lambda(B)/\,\mathrm{Sh}_{\Lambda\setminus\Omega}(B)$$

where B is the image of Ω under $S^*M \to M$.

Example 2.13. Consider an open ball B in $\mathbb{R}^2$ and let Λ be its outward conormal at the infinity $N^*_{out,\infty}(B)$, which has a homotopy type of an S^1. The category $\mu\mathrm{sh}_\Lambda(\Lambda)$ is then in fact the same as $\mathrm{Loc}(S^1)$. This is because locally near the front projection ∂B, a microsheaf can be represented by some constant sheaf with stalk A supported on an open half-plane but there can be monodromy when goin around the circle.



Because $\mu\mathrm{sh}_{T^*M;\Lambda}$ is a sheaf, the inclusion $q : \dot{T}^*M \subseteq T^*M$ induces a canonical restriction map

$$q^* : \mathrm{Sh}_\Lambda(M) \to \mu\mathrm{sh}_\Lambda(\Lambda),$$

often referred as the microlocalization functor, and we can see from the last example is neither fully-faithful nor surjective.

Example 2.14. This is the A-side functor corresponding to $i^! : \mathrm{IndCoh}(X_\Sigma) \to \mathrm{IndCoh}(\partial X_\Sigma)$ when setting $M = T^n$ and $\Lambda = \Lambda_\Sigma$ the FLTZ skeleton.

A main property of $q^* : \mathrm{Sh}_\Lambda(M) \to \mu\mathrm{sh}_\Lambda(\Lambda)$ is that it preserves both limits and colimits and thus admits both adjoints $q_! \dashv q^* \dashv q_*$. This fits it into the framework of spherical adjunctions.

Definition 2.15. Let $F : \mathcal{A} \leftrightarrows \mathcal{B} : F^L$ be an adjunction. We use the notation T and S to denote the functors which fit in the fiber sequences

$$T \to \mathrm{id}_\mathcal{B} \to FF^L,\ F^LF \to \mathrm{id}_\mathcal{A} \to S$$

and call them the twist and cotwist. The adjunction is called spherical if both T and S are equivalences.

2.4 Wrappings

We will give a description of the cotwist of q^* in terms of isotopies, or sometimes referred as wrappings, of sheaves. That is, for an isotopy $\varphi_t : S^*M \to S^*M$ for $t \in I$, we would like to construct, for a sheaf $F \in \mathrm{Sh}(M)$, a family $F_t \in \mathrm{Sh}(M)$ such that $\mathrm{SS}^\infty(F_t) = \varphi_t\left(\mathrm{SS}^\infty(F)\right)$. This is provided by the following theorem:

Theorem 2.16 ([3, Proposition 3.2, Theorem 3.7])**.** *Let M be a manifold. For a contact isotopy $\Phi : S^*M \times I \to S^*M$, there exists a unique sheaf kernel $K(\Phi) \in \mathrm{Sh}(M \times M \times I)$ such that*

1. $K(\Phi)|_{t=0} = 1_{\Delta_M}$, *and*
2. $\mathrm{SS}^\infty(K(\Phi)) \subseteq \Lambda_\Phi$ *where* $\Lambda_\Phi = \{(x, -\xi, \varphi_t(x,\xi), t, -\alpha(\dot{\varphi}_t))\}$ *is the contact movie of* Φ.

Moreover, this quantization is compatible with composition, i.e.,

1. $K(\Psi \circ \Phi) = K(\Psi) \circ |_I K(\Phi)$,
2. $K(\Phi^{-1}) \circ |_I K(\Phi) = K(\Phi) \circ |_I K(\Phi^{-1}) = 1_{\Delta_M \times I}$.

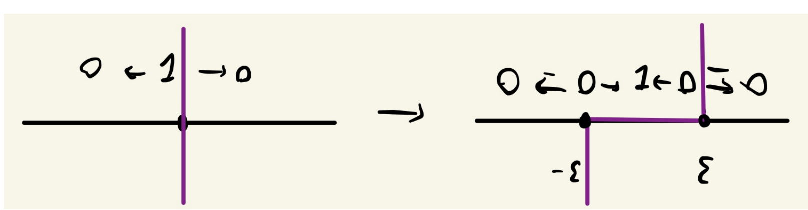
Here Φ^{-1} is the isotopy given by $\Phi^{-1}(-,t) \coloneqq \varphi_t^{-1}$.

With the above we theorem, we obtain the family by setting $F_t \coloneqq \left(K(\Phi) \circ F\right)|_t = K(\Phi)|_t \circ F$. In fact, considering the total sheaf $K(\Phi) \circ F \in \mathrm{Sh}(M \times I)$ provides some more structure for us.

Example 2.17. The simplest example of wrapping is given by the isotopy $\Phi : T^*J \times I \to T^*J$ where $J = \mathbb{R}^1$ or S^1 by the formula

$$\varphi_t(x,\xi) = \begin{cases} (x+t,\xi),\ \xi > 0, \\ (x-t,\xi),\ \xi < 0. \end{cases}$$

This is the case where the term "wrapping" comes from. For the $\mathbb{R}^1$ case, when $t > 0$, the GKS sheaf quantization is simply $1_{\{(x,y)||x-y|<t\}}[1]$ and when $t \leq 0$, it is given by $1_{\{(x,y)||x-y|\leq -t\}}$. The S^1 are given by a suitable projection of them.



Definition 2.18. We say a contact isotopy $\Phi : S^*M \times I \to S^*M$ is positive if $\alpha(\dot{\varphi}_t) \geq 0$.

For such an isotopy, we see from the description of the contact movie Λ that $\mathrm{SS}\left(K(\Phi)\right) \subseteq \{\tau \leq 0\}$, i.e., $K(\Phi) \in \mathrm{Sh}_{\{\tau \leq 0\}}(M \times M \times I)$. For any manifold N, a sheaf $G \in \mathrm{Sh}_{\{\tau \leq 0\}}(N \times I)$ admits continuation maps, they are a family of maps

$$c_{s,t} : G|_s \to G|_t$$

for any $s \leq t$ and they composes naturally, for example, $c_{r,t} \circ c_{s,t} = c_{s,r}$. Thus, for a positive isotopy Φ and a sheaf F, we have continuation maps $F \to F_t$ for $t \geq 0$. To simplify the notation when varying the isotopies, we also use the notation $F^\varphi := K(\Phi)|_1 \circ F$ for a given isotopy.

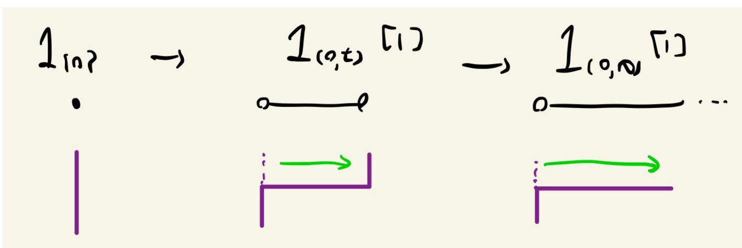
Proposition 2.19 ([6, Theorem 1.2])**.** *Let $X \subseteq S^*M$ be a closed subset. The left adjoint ι^* of the inclusion $\iota_* : \mathrm{Sh}_X(M) \hookrightarrow \mathrm{Sh}(M)$ admits the description*

$$\mathfrak{W}_X^+(F) := \operatorname*{colim}_{\Phi : X^c} F^\varphi$$

where we runs through positive contact isotopy Φ which are compactly supported away from X. There is a similar description of the right adjoint by negative wrapping.

Example 2.20. Let $M = \mathbb{R}^1$ and $\Lambda = (0, -1)$. The sheaf $1_{\{0\}} \in \mathrm{Sh}(\mathbb{R}^1)$ satisfies $\mathrm{SS}(1_{\{0\}}) \cap \{\xi < 0\} \subseteq \Lambda$ but not the positive codirection. According to the above theorem, it $\iota^*(1_{\{0\}})$ can thus be computed using Example 2.17 with only wrapping on the positive end, which implies that

$$\iota^*(1_{\{0\}}) = \mathrm{colim}_{t\to\infty} 1_{(0,t)}[1] = 1_{(0,\infty)}[1].$$



Proposition 2.21 ([7, Remark 4.4, Definition 4.5])**.** *Fix a small positive contact isotopy φ_t such that $\Lambda \cap \varphi_t(\Lambda) = \varnothing$ for $0 < t < \epsilon$. Then we have $S_\Lambda^+(F) = \mathfrak{W}_\Lambda^+(F^\varphi)$.*

One thing this description provides is, in good cases, a description of the Serre functor on the subcategory of sheaves with perfect stalks and compact support $\mathrm{Sh}_\Lambda(M)_0^b \subseteq \mathrm{Sh}_\Lambda(M)^c$.

Corollary 2.22 ([7, Proposition 5.28])**.** *For a swappable $\Lambda \subseteq S^*M$ [7, Proposition 5.19], the Serre functor $\mathcal{S}\mathrm{r}$, the unique functor such that,*

$$\mathrm{Hom}(G, F)^\vee = \mathrm{Hom}(F, \mathcal{S}\mathrm{r}(G))$$

on $\mathrm{Sh}_\Lambda(M)_0^b$ is given by $\mathcal{S}\mathrm{r}(F) = S_\Lambda^-(F \otimes \omega_M)$.

Assume the manifold M is compact. Then we have $\mathrm{Sh}_\Lambda(M)^b \subseteq \mathrm{Sh}_\Lambda(M)^c$, i.e., sheaves with perfect stalks are compact. The (classical) Verdier duality

$$D_M : \mathrm{Sh}(M) \to \mathrm{Sh}(M)^{op}$$
$$F \mapsto \mathcal{H}\mathrm{om}(F, \omega_M)$$

restricts to an equivalence $D_M : \mathrm{Sh}_\Lambda(M)^{b,op} \xrightarrow{\sim} \mathrm{Sh}_{-\Lambda}(M)^b$. On the other hand, there is a Fourier-Mukai theorem

$$\mathrm{Sh}_{-\Lambda\times\Sigma}(M \times N) \xrightarrow{\sim} \mathrm{Fun}^L(\mathrm{Sh}_\Lambda(M), \mathrm{Sh}_\Sigma(N))$$
$$F \mapsto K \circ F$$

and it follows from a canonical duality $D_\Lambda : \mathrm{Sh}_{-\Lambda}(M)^c \xrightarrow{\sim} \mathrm{Sh}_\Lambda(M)^{c,op}$.

Proposition 2.23 ([7, Proposition 7.19]). *For* $F \in \mathrm{Sh}_{-\Lambda}(M)^b$,

$$D_\Lambda(F) = \left(S_\Lambda^+\left(D_M(F)\right)\right) \otimes \omega_M^{-1}.$$

Remark 2.24. One sees easily that when S_Λ^+ is invertible, the equivalence $D_M : \mathrm{Sh}_\Lambda(M)^{b,op} \xrightarrow{\sim} \mathrm{Sh}_{-\Lambda}(M)^b$ extends to the whole $\mathrm{Sh}_{-\Lambda}(M)^c \xrightarrow{\sim} \mathrm{Sh}_\Lambda(M)^{c,op}$. In fact, the converse in some precise but somewhat complicated sense is also true. See [7, Theorem 7.22].

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